KECE321 Communication Systems I

(Haykin Sec. 2.6 and Ziemer Sec. 2.7)

Lecture #7, March 26, 2012

Prof. Young-Chai Ko

Review

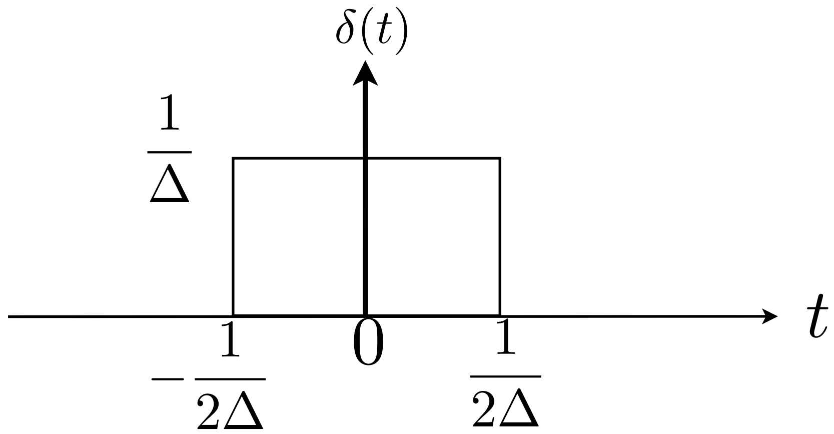
- Fourier transform

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} \, dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} \, dt$$

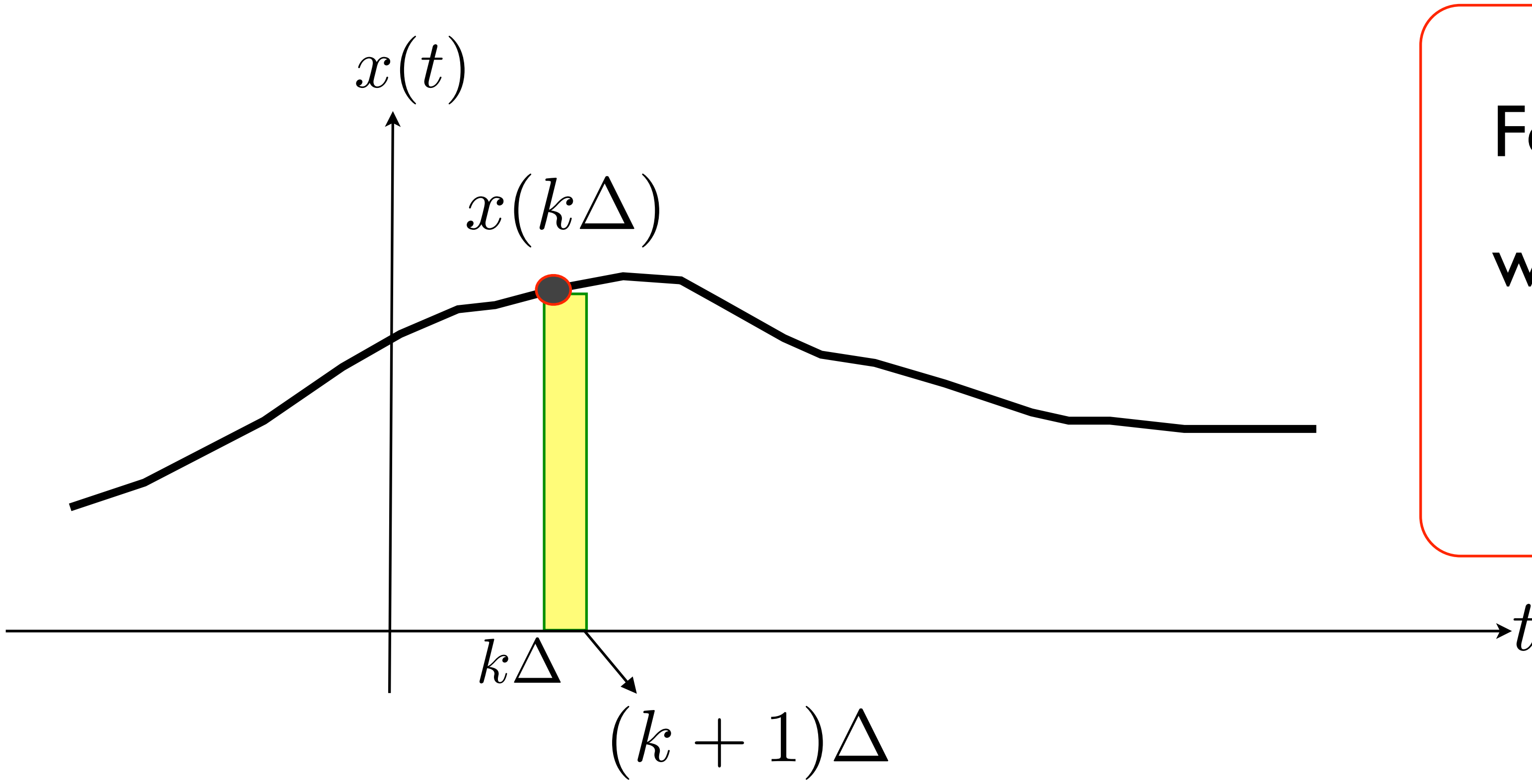
- Impulse function

$$x(t) = \int_{-\infty}^{\infty} x(\lambda) \delta(t - \lambda) \, d\lambda$$



$$\lim_{\epsilon \to 0} \frac{1}{\Delta} \mathrm{rect}\left(\frac{t}{\Delta}\right) = \delta(t)$$

$$\int_{-\infty}^{\infty} \delta(t) = 1 \Rightarrow \delta(t) \cdot \epsilon = 1$$

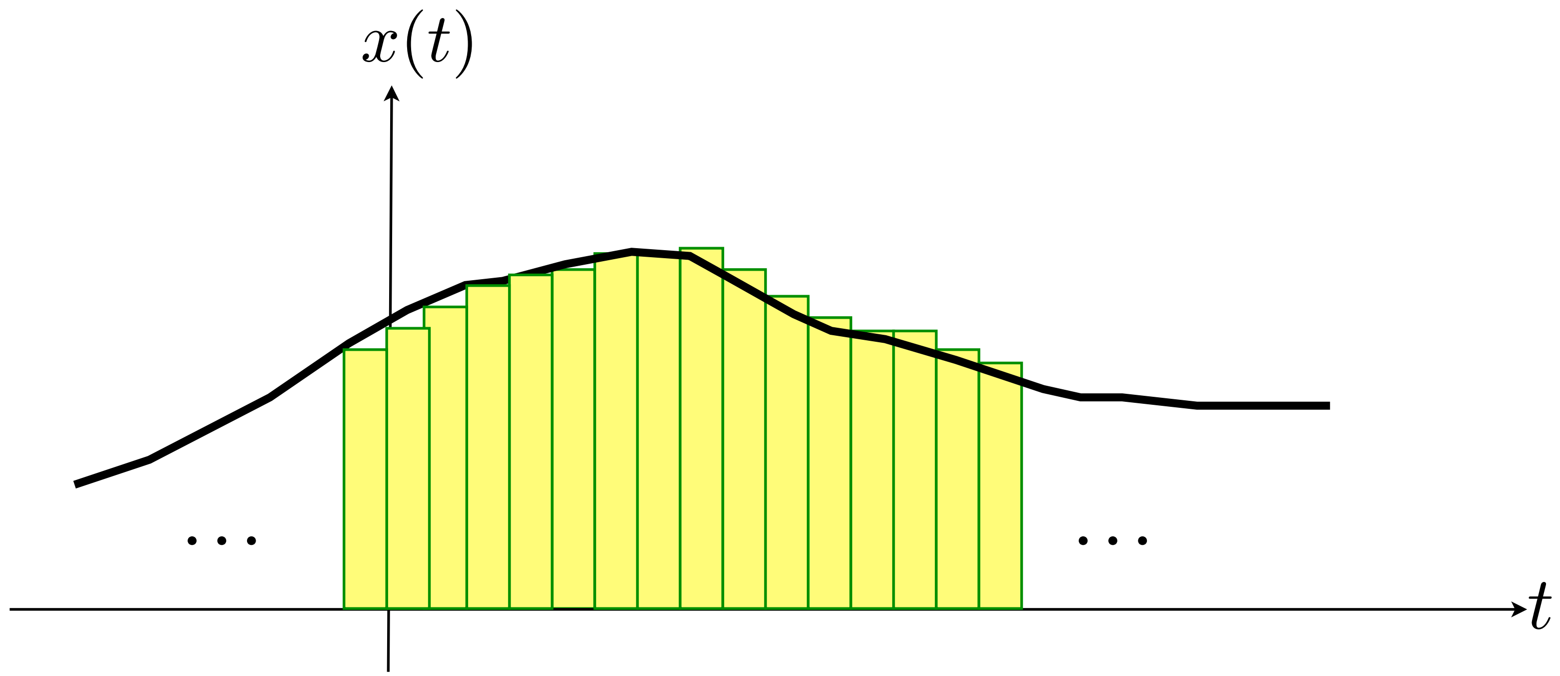


For $k\Delta \leq t < (k+1)\Delta$

we approximate $x(t)$ as

$$x(k\Delta) \cdot \delta(t - k\Delta)\Delta$$

for $\Delta \to 0$



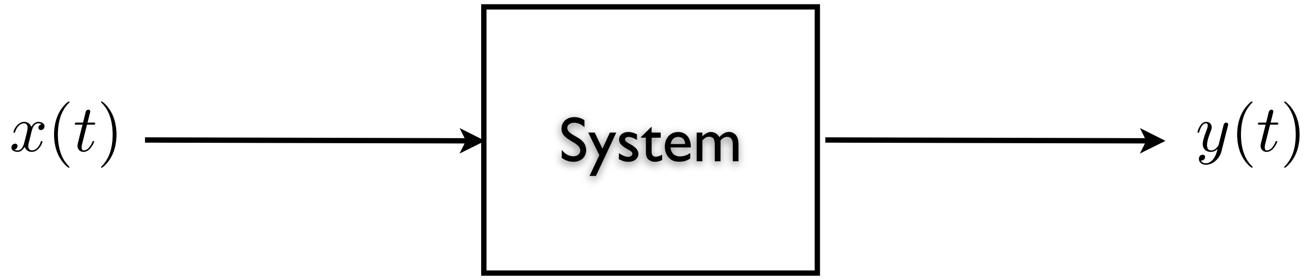
$$x(t) \approx \sum_{n=-\infty}^{\infty} x(n\Delta t)\delta(t - n\Delta t)\Delta t, \quad \Delta t << 1$$

Summary

- Linear Time-Invariant (LTI) Systems
  - Time response
  - Frequency response
- Convolution theorem
- Filter
  - Low-pass filter
  - Band-pass filter
  - High-pass filter

System Response

Time response



$$y(t) = \mathcal{H}[x(t)]$$

Frequency response

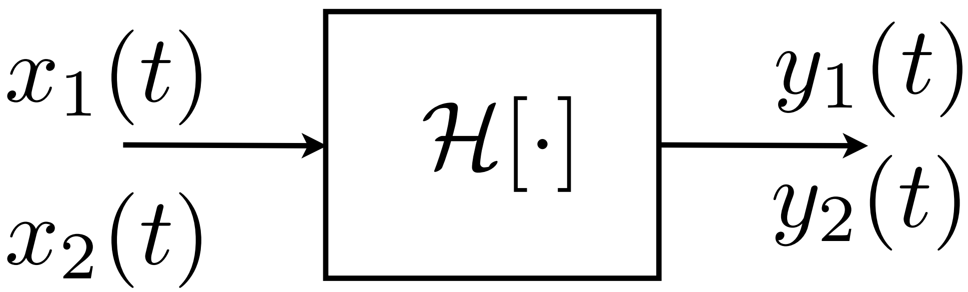
$$Y(f) = \mathcal{H}[X(f)]$$

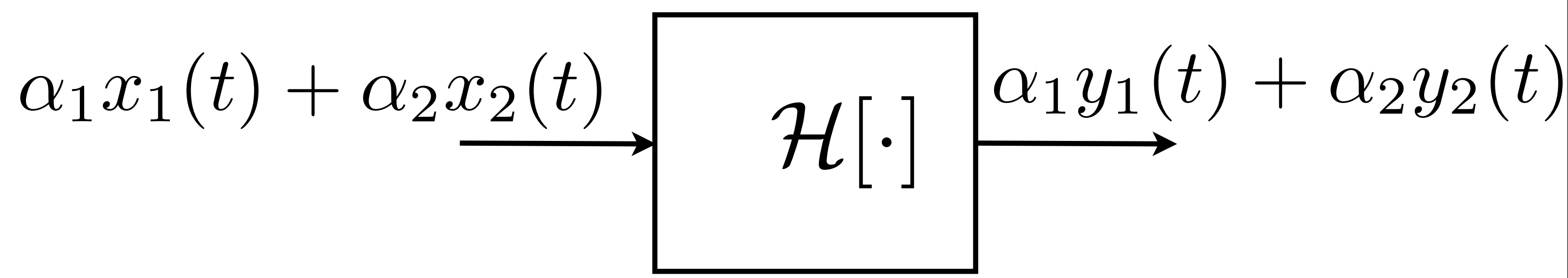
Linear Time-Invariant (LTI) System

- Linear system if

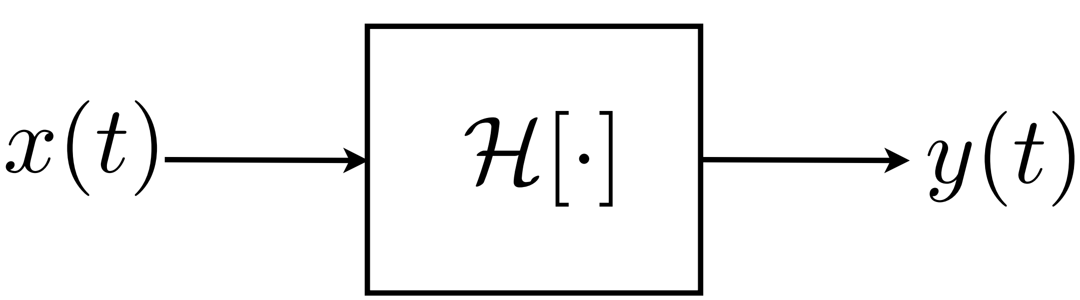
$$y(t) = \mathcal{H}[\alpha_1 x_1(t) + \alpha_2 x_2(t)] = \alpha_1 \mathcal{H}[x_1(t)] + \mathcal{H}[x_2(t)]$$

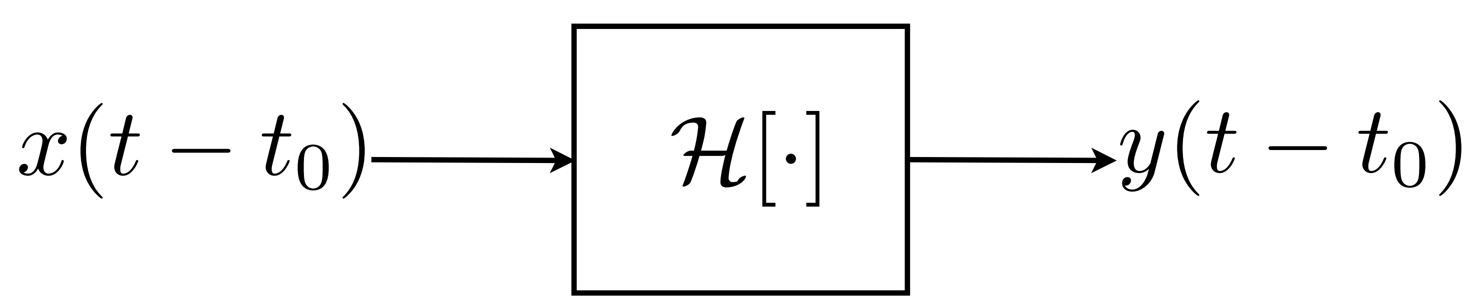
$$= \alpha_1 y_1(t) + \alpha_2 y_2(t)$$





- Time invariant system if $y(t - t_0) = \mathcal{H}[x(t - t_0)]$

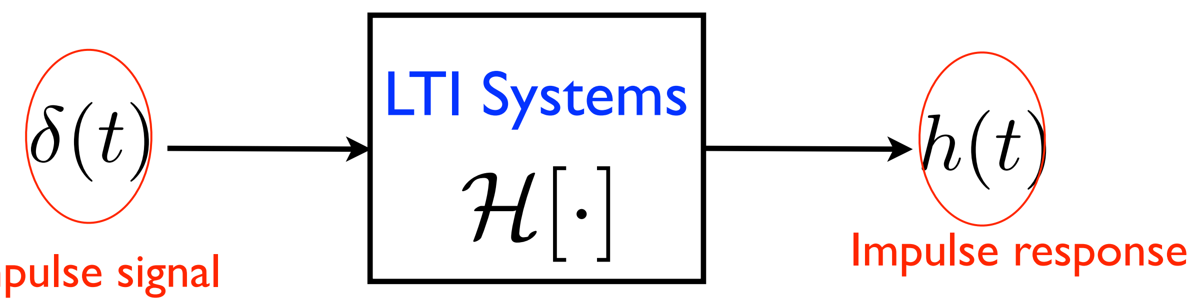




Impulse Response to LTI System

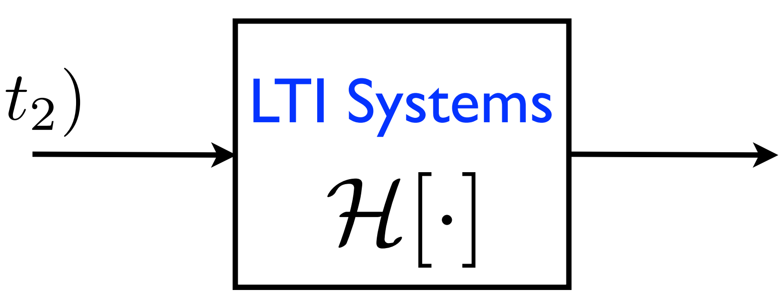
- Impulse response

$$h(t) \triangleq \mathcal{H}[\delta(t)]$$



- Response of linear sum of the impulse signals with time shift

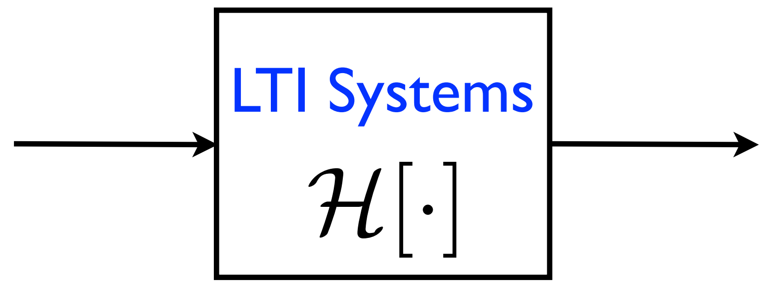
$$x(t) = \alpha_1 \delta(t - t_1) + \alpha_2 \delta(t - t_2)$$



$$y(t) = \alpha_1 h(t - t_1) + \alpha_2 h(t - t_2)$$

- In more general, we have the following relation in the LTI system:

$$x(t) = \sum_{n=1}^{N} \alpha_n \delta(t - t_n)$$



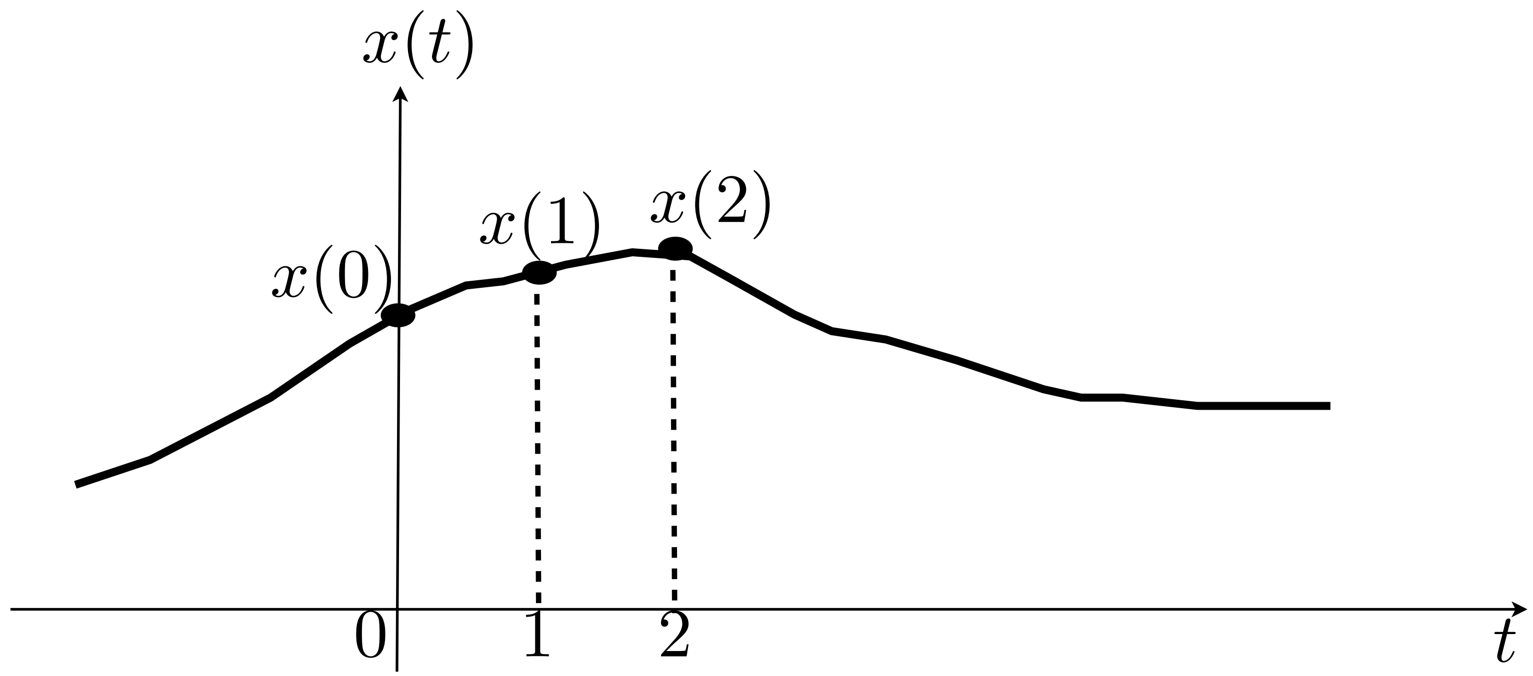
$$y(t) = \sum_{n=1}^{N} \alpha_n h(t - t_n)$$

- Recall the definition of the impulse response

$$x(t) = \int_{-\infty}^{\infty} x(\lambda)\delta(t - \lambda)\, d\lambda$$

Approximation the integral as a sum:

$$x(t) \approx \sum_{n=-\infty}^{\infty} x(n\Delta t)\delta(t - n\Delta t)\Delta t, \quad \Delta t << 1$$

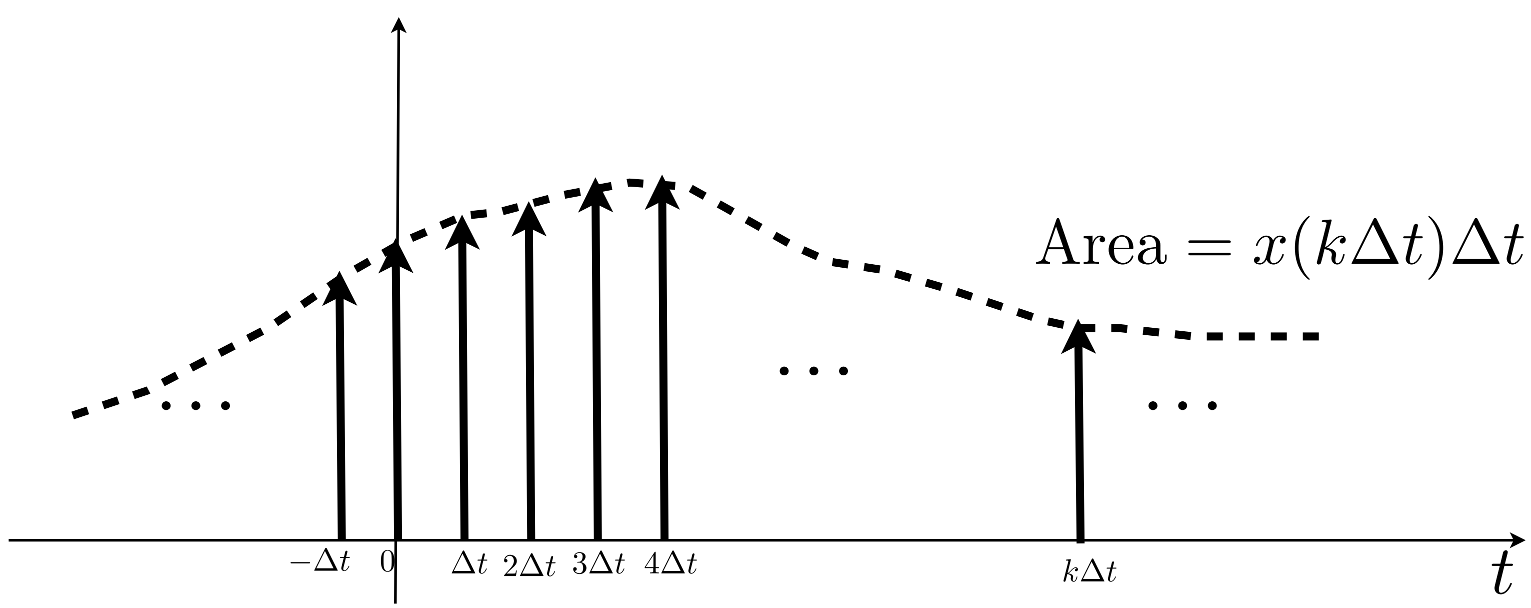


$$x(0) = \int_{-\infty}^{\infty} x(\lambda)\delta(\lambda)\, d\lambda$$

$$x(1) = \int_{-\infty}^{\infty} x(\lambda)\delta(1-\lambda)\, d\lambda$$

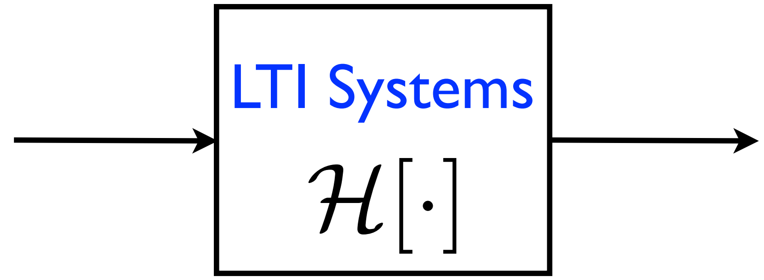
$$x(2) = \int_{-\infty}^{\infty} x(\lambda)\delta(2-\lambda)\, d\lambda$$

$$\vdots$$



$$x(t) \approx \sum_{n=-\infty}^{\infty} x(n\Delta t)\delta(t - n\Delta t)\Delta t, \quad \Delta t << 1$$

$$\tilde{x}(t) \approx \sum_{n=-\infty}^{\infty} x(n\Delta t)\delta(t - n\Delta t)\Delta t$$



$$\tilde{y}(t) = \sum_{n=-\infty}^{\infty} x(n\Delta t)h(t - n\Delta t)\Delta t$$

$$y(t) = \lim_{\Delta t \to 0} \tilde{y}(t) = \lim_{\Delta t \to 0} \sum_{n=-\infty}^{\infty} x(n\Delta t)h(t - n\Delta t)\Delta t$$

λ $\quad$ λ $\quad$ $d\lambda$

$$\int_{-\infty}^{\infty}$$

$$= \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)\, d\lambda$$

By change of variable such as $t' = t - \lambda$, we can show that

$$y(t) = \int_{-\infty}^{\infty} x(t-\lambda)h(\lambda)\, d\lambda$$

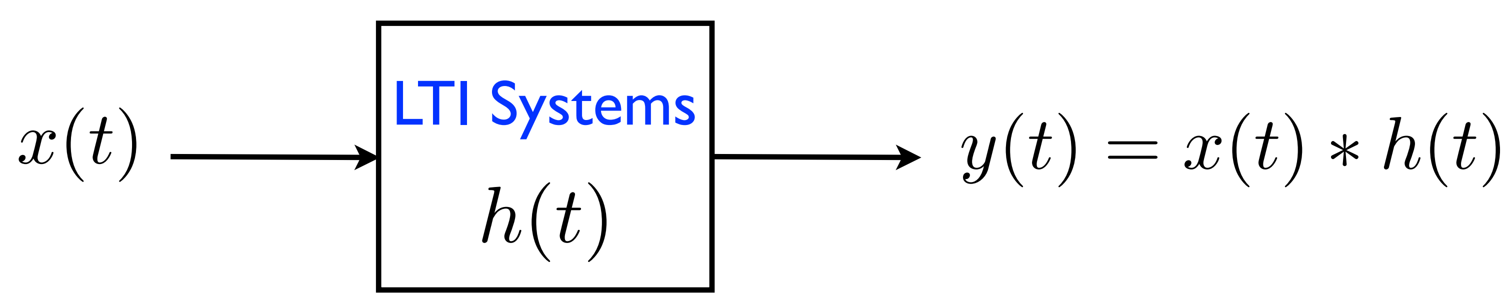
Convolution

- Definition of Convolution

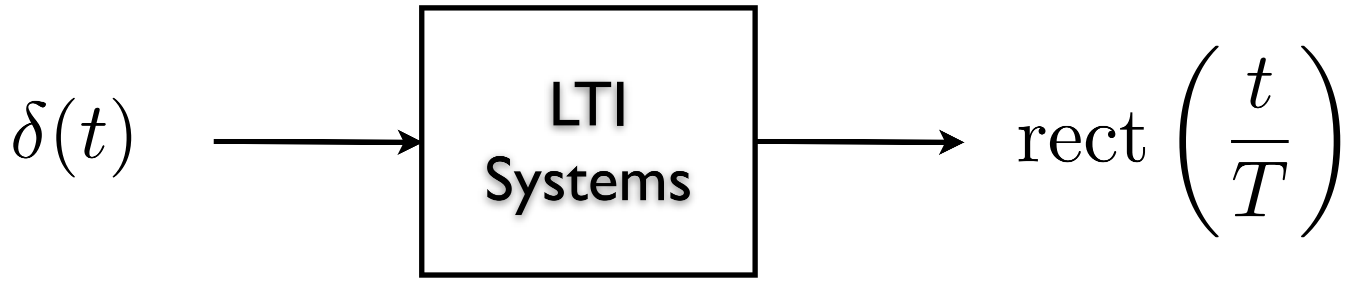
$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)\,d\lambda$$

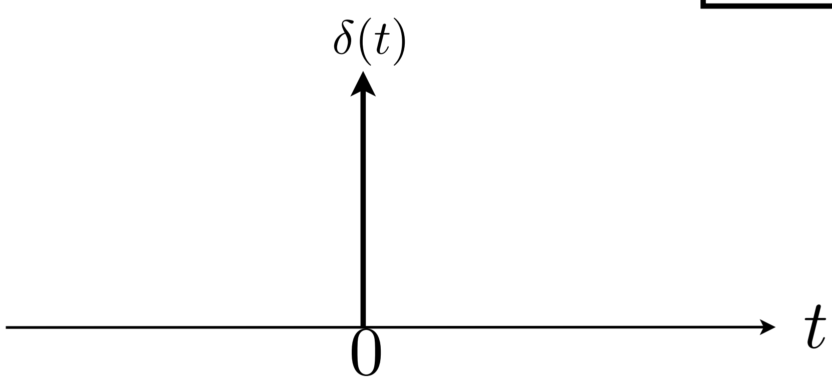
$$= \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)\,d\lambda$$

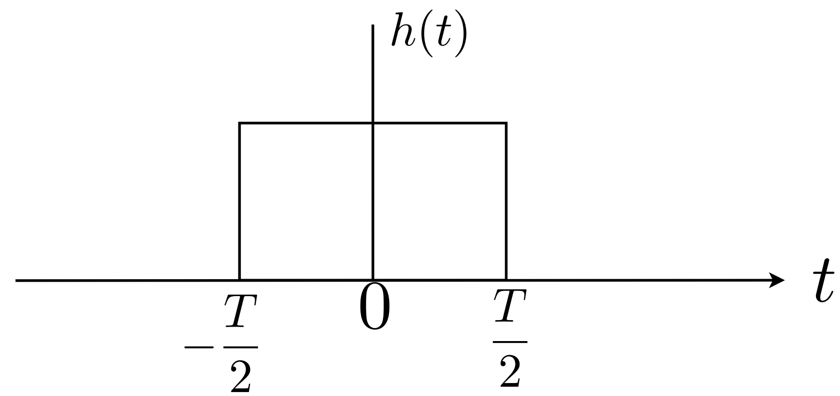
- Signal response by convolution operation



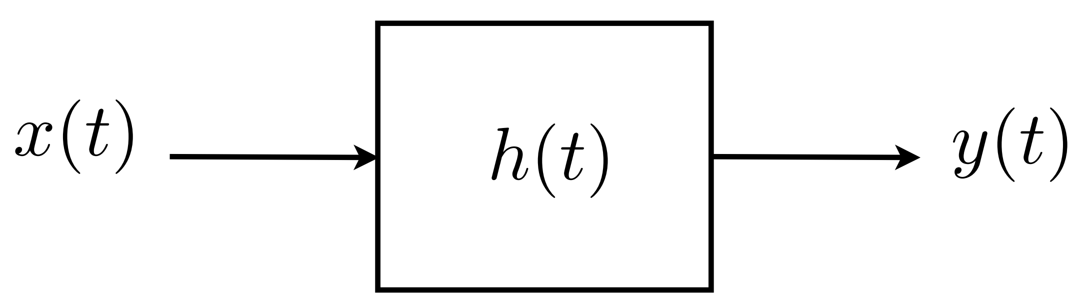
Example:



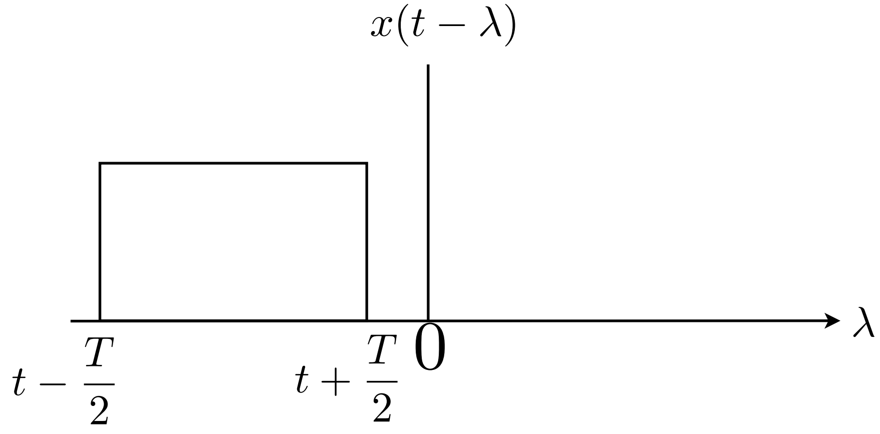


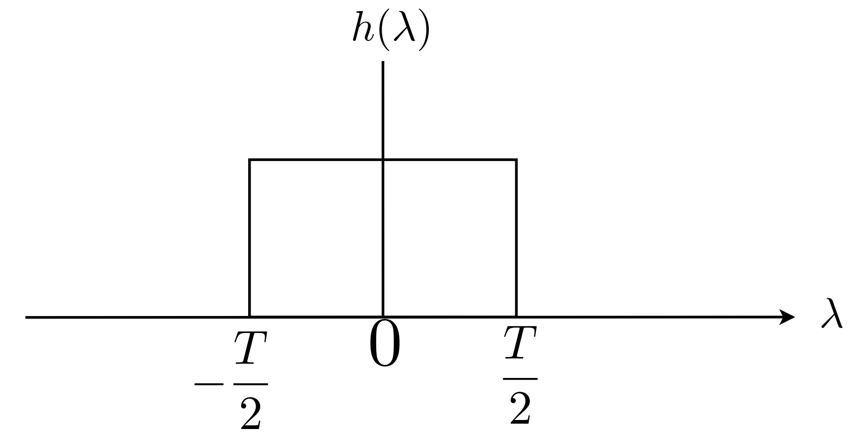


What is the output response for the input signal $x(t) = \text{rect}\left(\frac{t}{T}\right)$?



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\lambda) x(t-\lambda)\, d\lambda$$



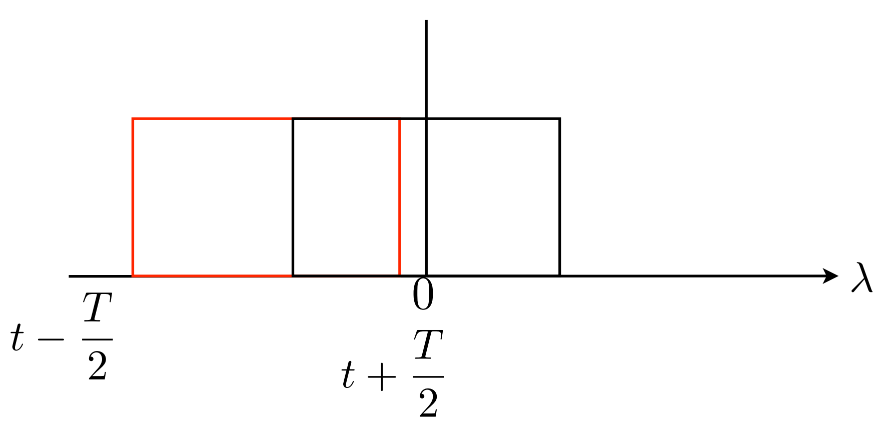


$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)\, d\lambda$$

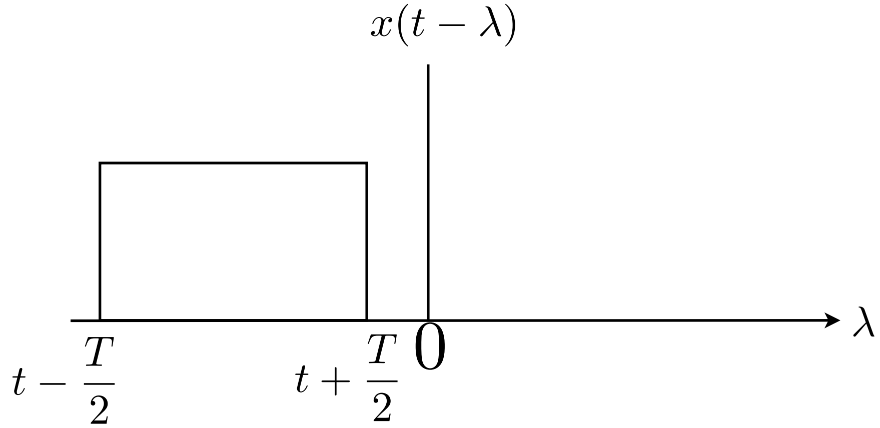
1) If $t + \frac{T}{2} < -\frac{T}{2}$ **, that is,** $t < -T$

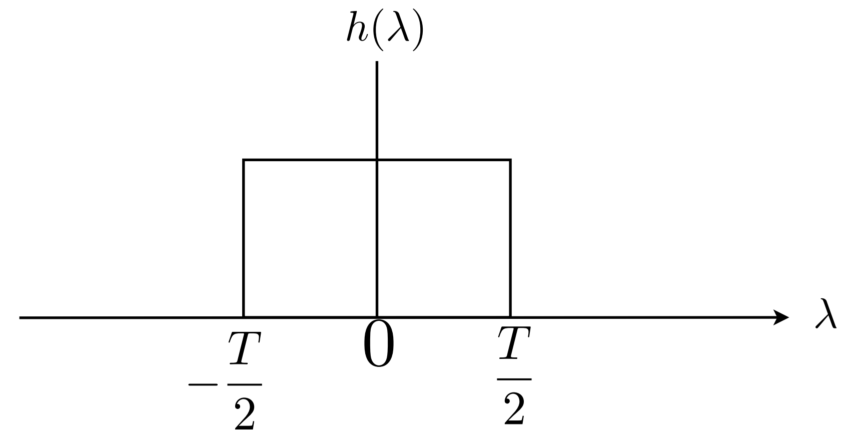
$$y(t) = 0$$

2) If $-\frac{T}{2} \leq t + \frac{T}{2} < \frac{T}{2}$**, that is,** $-T \leq t < 0$

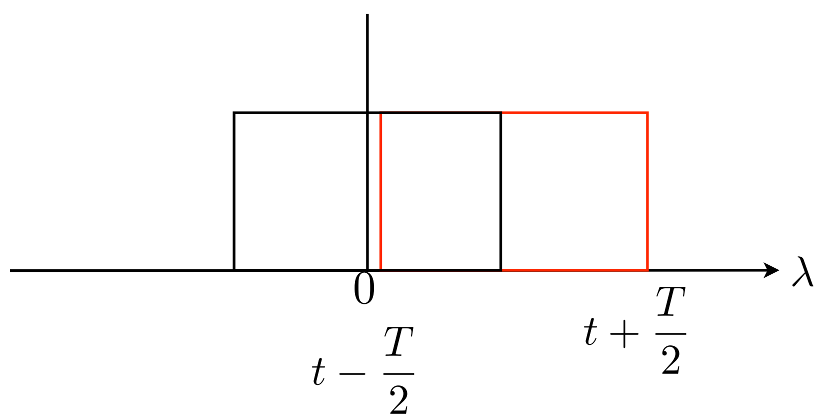


$$y(t) = \int_{-\frac{T}{2}}^{t+\frac{T}{2}} \; dt \quad = t + T$$



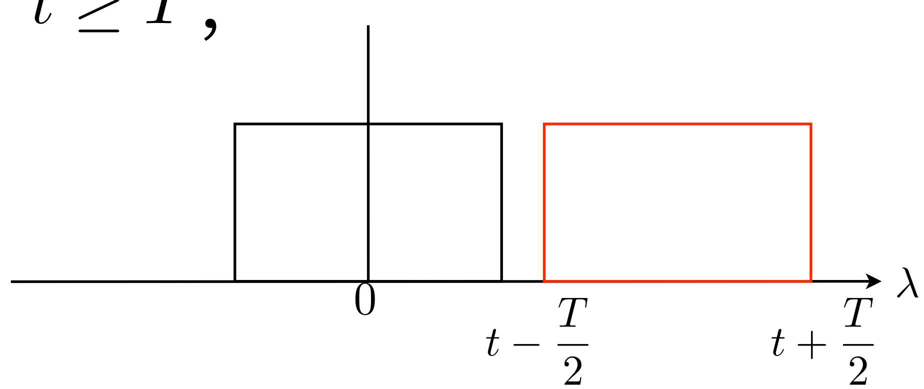


3) If $-\frac{T}{2} \leq t - \frac{T}{2} < \frac{T}{2}$, that is, $0 \leq t < T$



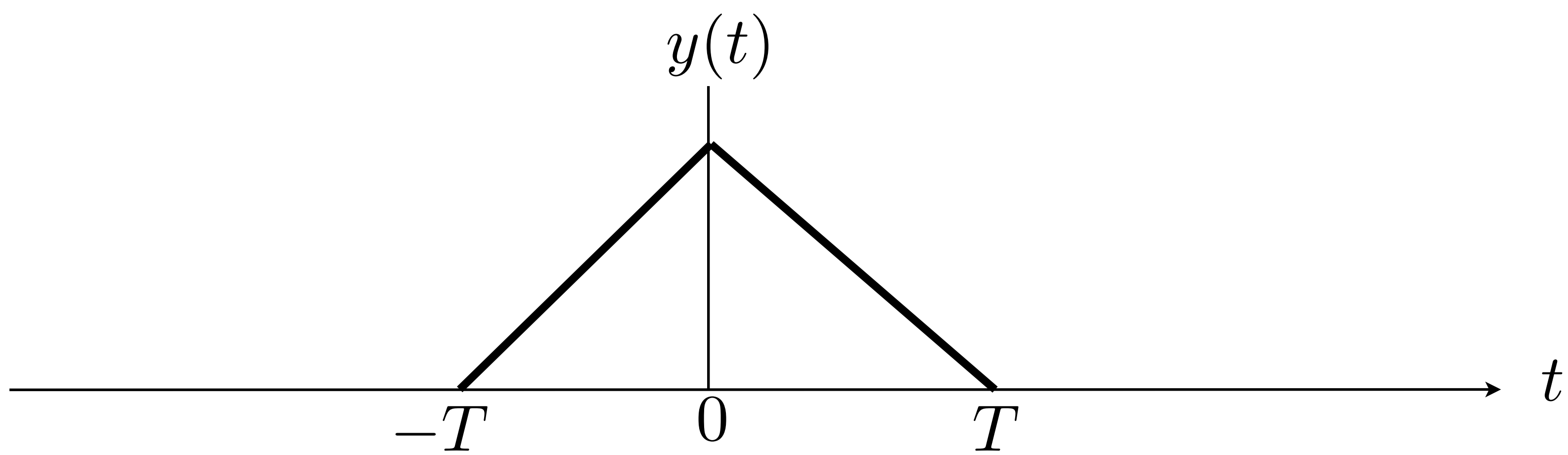
$$y(t) = \int_{t-\frac{T}{2}}^{\frac{T}{2}} dt \quad = T - t$$

4) If $t \geq T$,



$$y(t) = 0$$

$$y(t) = \begin{cases} t + T, & -T \leq t < 0 \\ t - T, & 0 \leq t < T \\ 0, & \text{otherwise} \end{cases}$$



Convolution Theorem in Fourier Transform

- Convolution theorem

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda) x_2(t-\lambda)\, d\lambda = \int_{-\infty}^{\infty} x_1(t-\lambda) x_2(\lambda)\, d\lambda$$

$$\xleftrightarrow{\mathcal{F}[\cdot]} \quad X_1(f) X_2(f)$$

Proof:

$$x_2(t-\lambda) = \int_{-\infty}^{\infty} X_2(f) e^{j2\pi f(t-\lambda)}\, d\lambda$$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda) x_2(t-\lambda)\, d\lambda$$

$$= \int_{-\infty}^{\infty} x_1(\lambda) \left[\int_{-\infty}^{\infty} X_2(f) e^{j2\pi f(t-\lambda)}\, df \right] d\lambda$$

$$= \int_{-\infty}^{\infty} X_2(f) \left[\int_{-\infty}^{\infty} x_1(\lambda) e^{-j2\pi f\lambda}\, d\lambda \right] e^{j2\pi f t}\, df$$

$X_1(f)$

$$= \int_{-\infty}^{\infty} X_1(f) X_2(f) e^{j2\pi f t}\, df \quad = \mathcal{F}^{-1}[X_1(f) * X_2(f)]$$

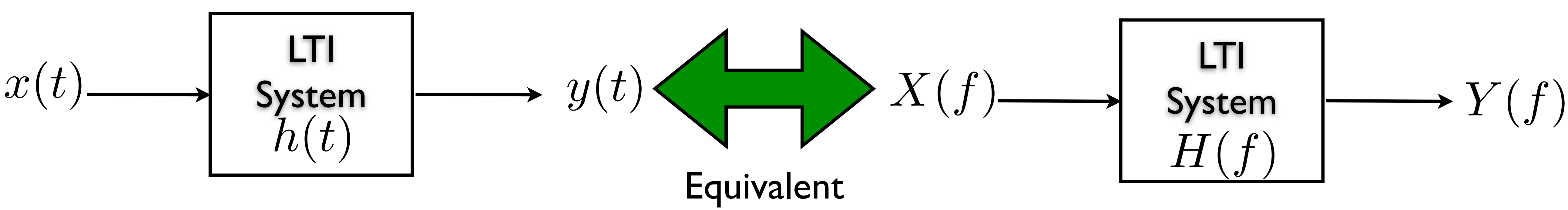
Frequency Response in LTI Systems

- Frequency response

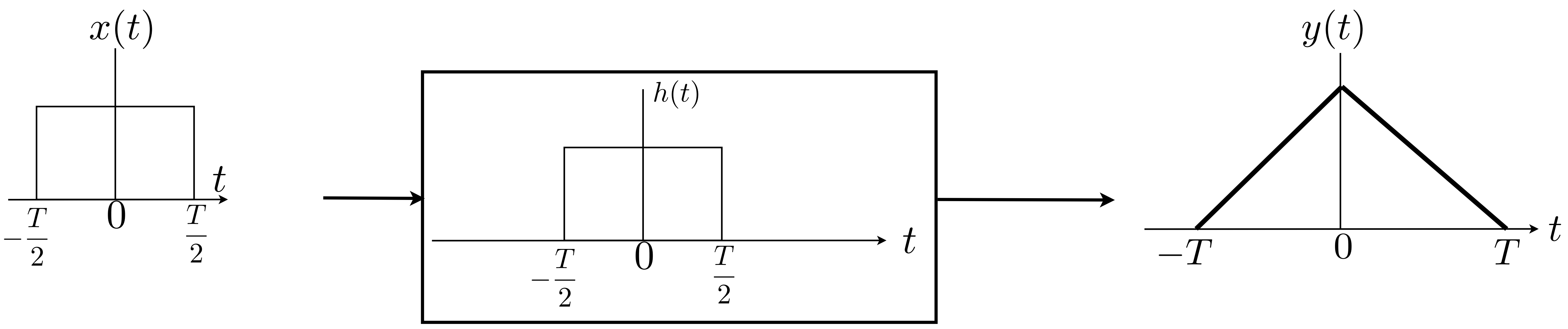
$$H(f) = \mathcal{F}[h(t)]$$

- Using the convolution theorem, we have Y(f)

$$x(t) * h(t) \xrightarrow{\mathcal{F}[\cdot]} X(f)H(f)$$



Example:



Find $Y(f)$.

Sol:

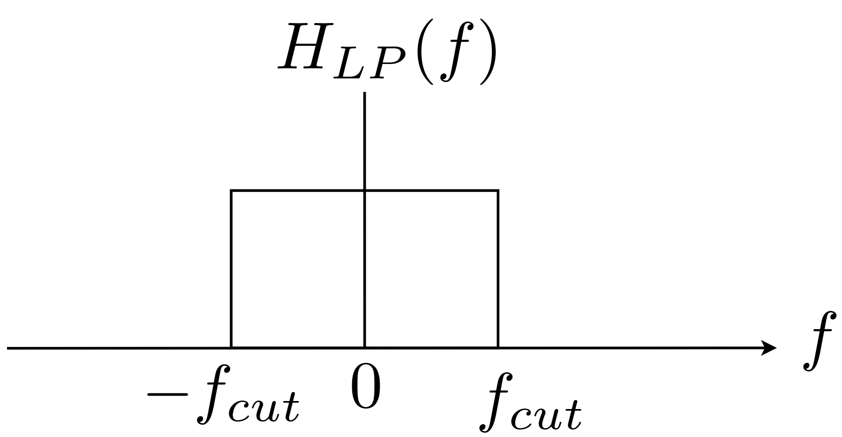
$$X(f) = H(f) = T\text{sinc}(Tf)$$

Hence, we have

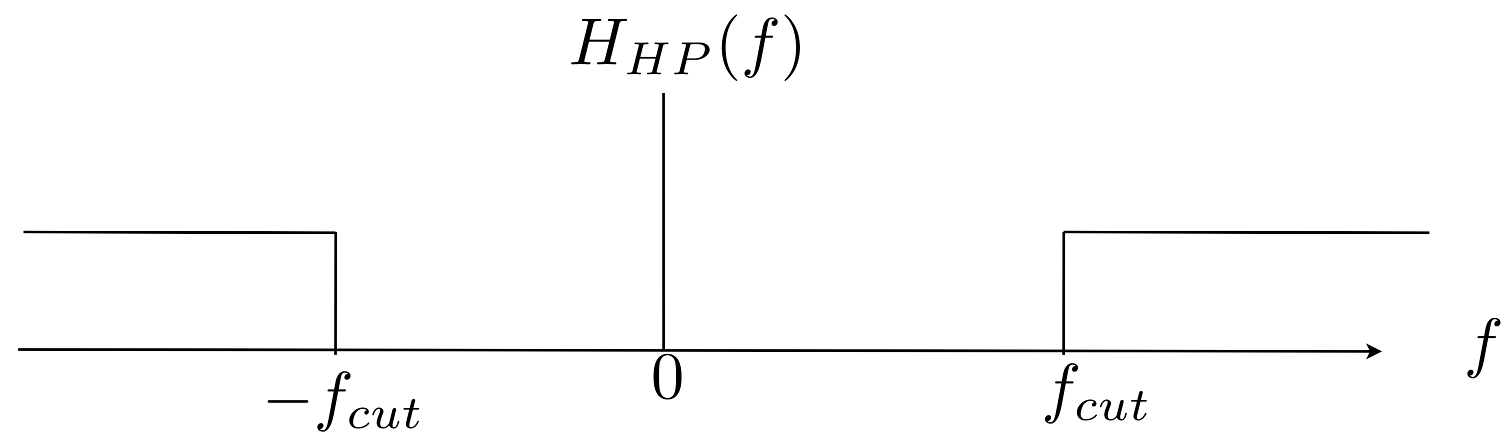
$$Y(f) = X(f)H(f) = \left(T\text{sinc}(TF)\right)^2$$

Low-Pass/Pass-Band/High-Pass Filter

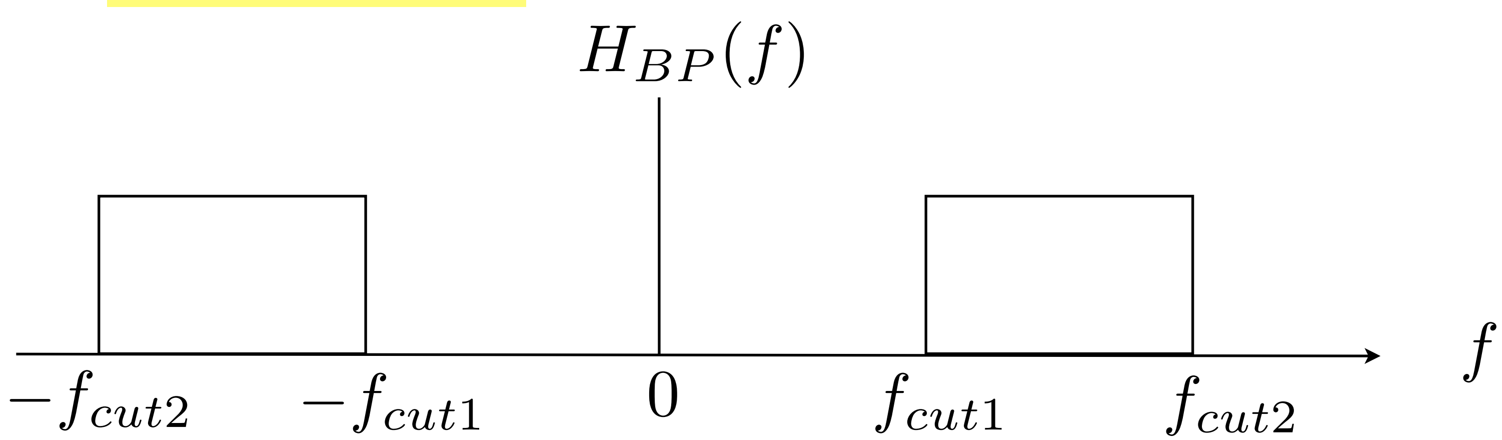
Low-pass filter



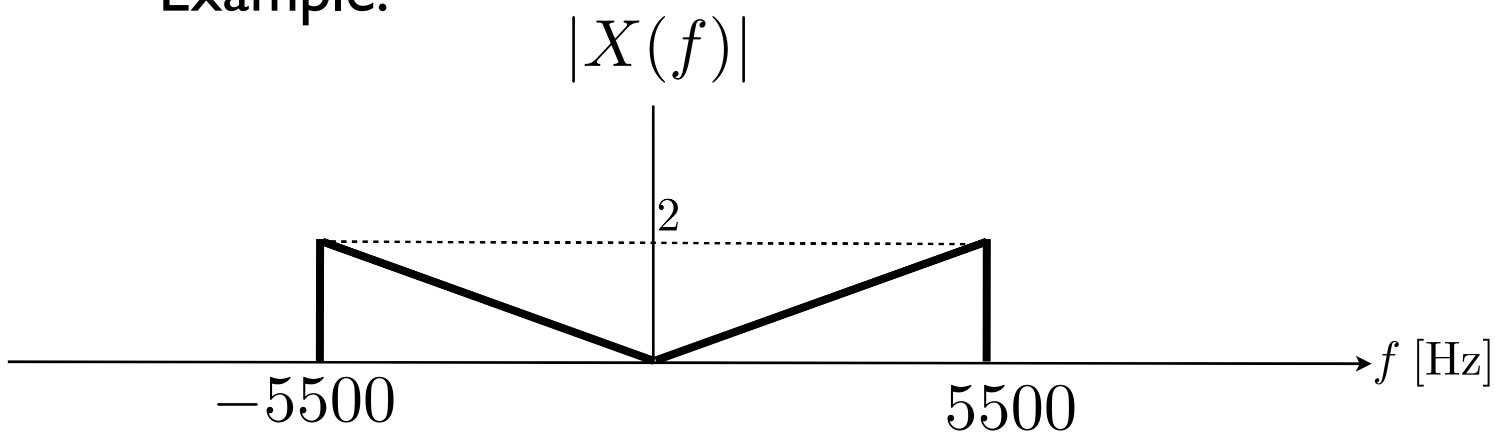
High-pass filter

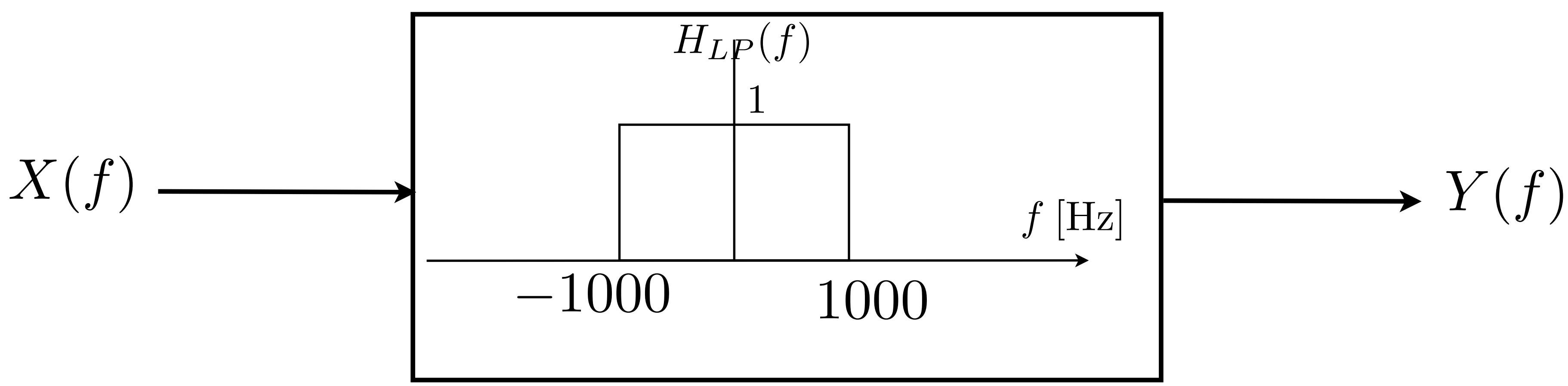


Pass-band filter



Example:





Draw the amplitude spectrum of $Y(f)$.