Numerical Analysis MTH614

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Finite Differences for the Heat Equation 2

Thomas algorithm

In this lecture, we consider the method of solving in an efficient manner. The key fact that allows us to do so is the tridiagonal form of A. Let's see A has nonzero entries only on the main diagonal and on the subdiagonals directly above and below them.

$$
\begin{pmatrix}
d_1 & c_1 & & & & & & \\
a_1 & d_2 & c_2 & & & & & \\
 & a_2 & d_3 & c_3 & & & & \\
 & & \ddots & \ddots & \ddots & & & \\
 & & & a_{i-1} & d_i & c_i & & \\
 & & & & \ddots & \ddots & \ddots & \\
 & & & & & a_{N_x-2} & d_{N_x-1} & c_{N_x-1} \\
 & & & & & & a_{N_x-1} & d_{N_x}
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_{N_x-1} \\ x_{N_x}
\end{pmatrix}
=
\begin{pmatrix}
b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_i \\ \vdots \\ b_{N_x-1} \\ b_{N_x}
\end{pmatrix},
$$

We do that $d_2 = d_2 - \frac{a_1}{d_1} c_1, \qquad b_2 = b_2 - \frac{a_1}{d_1} b_1$ (1)

But never change c_2

$$
\begin{pmatrix}
d_1 & c_1 & & & & & & \\
a_1 & d_2 & c_2 & & & & & \\
& a_2 & d_3 & c_3 & & & & \\
& & \ddots & \ddots & \ddots & & & \\
& & & a_{i-1} & d_i & c_i & & \\
& & & & \ddots & \ddots & \ddots & \\
& & & & & a_{N_x-2} & d_{N_x-1} & c_{N_x-1} \\
& & & & & & a_{N_x-1} & d_{N_x}
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_{N_x-1} \\ x_{N_x}
\end{pmatrix}
=
\begin{pmatrix}
b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_i \\ \vdots \\ b_{N_x-1} \\ b_{N_x}
\end{pmatrix},
$$

$$
\begin{pmatrix}
d_1 & c_1 & & & & & & \\
0 & d_2 - \frac{a_1 c_1}{d_1} & c_2 & & & & & \\
 & a_2 & d_3 & c_3 & & & & \\
 & & \ddots & \ddots & \ddots & & & \\
 & & & a_{i-1} & d_i & c_i & & \\
 & & & & \ddots & \ddots & \ddots & \\
 & & & & & a_{N_x-2} & d_{N_x-1} & c_{N_x-1} \\
 & & & & & & a_{N_x-1} & d_{N_x}
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_{N_x-1} \\ x_{N_x}
\end{pmatrix}
=
\begin{pmatrix}
b_1 \\ b_2 - \frac{a_1 b_1}{d_1} \\ b_3 \\ \vdots \\ b_i \\ \vdots \\ b_{N_x-1} \\ b_{N_x}
\end{pmatrix}. \qquad (1)
$$

We repeat this process such that the relation of (1) until i reaches N_x,

$$
d_i = d_i - \frac{a_{i-1}}{d_{i-1}} c_{i-1}, \qquad b_i = b_i - \frac{a_{i-1}}{d_{i-1}} b_{i-1} \qquad (2 \le i \le N_x). \qquad (2)
$$

$$
\begin{pmatrix}
d_1 & c_1 & & & & & & \\
& d_2 & c_2 & & & & & \\
& & d_3 & c_3 & & & & \\
& & & \ddots & \ddots & & & \\
& & & & d_i & c_i & & \\
& & & & & \ddots & \ddots & \\
& & & & & & d_{N_x-1} & c_{N_x-1} \\
& & & & & & & d_{N_x}
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_{N_x-1} \\ x_{N_x}
\end{pmatrix}
=
\begin{pmatrix}
b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_i \\ \vdots \\ b_{N_x-1} \\ b_{N_x}
\end{pmatrix}. \tag{1}
$$

$$
\begin{aligned}
x_{N_x} &= \frac{b_{N_x}}{d_{N_x}}, \\
x_i &= \frac{1}{d_i}(b_i - c_i x_{i+1}), \qquad i = N_x - 1,\ N_x - 2,\ \cdots,\ 1.
\end{aligned} \tag{2}
$$

We consider one dimensional problem such as

$$
\partial_t u(x,t) = u_{xx}(x,t),\ 0 < x < 1,\ t > 0. \tag{3}
$$

Its boundary and initial condition are given

$$
u(0,t) = u(1,t) = 0,\ u(x,0) = \sin(\pi x). \tag{4}
$$

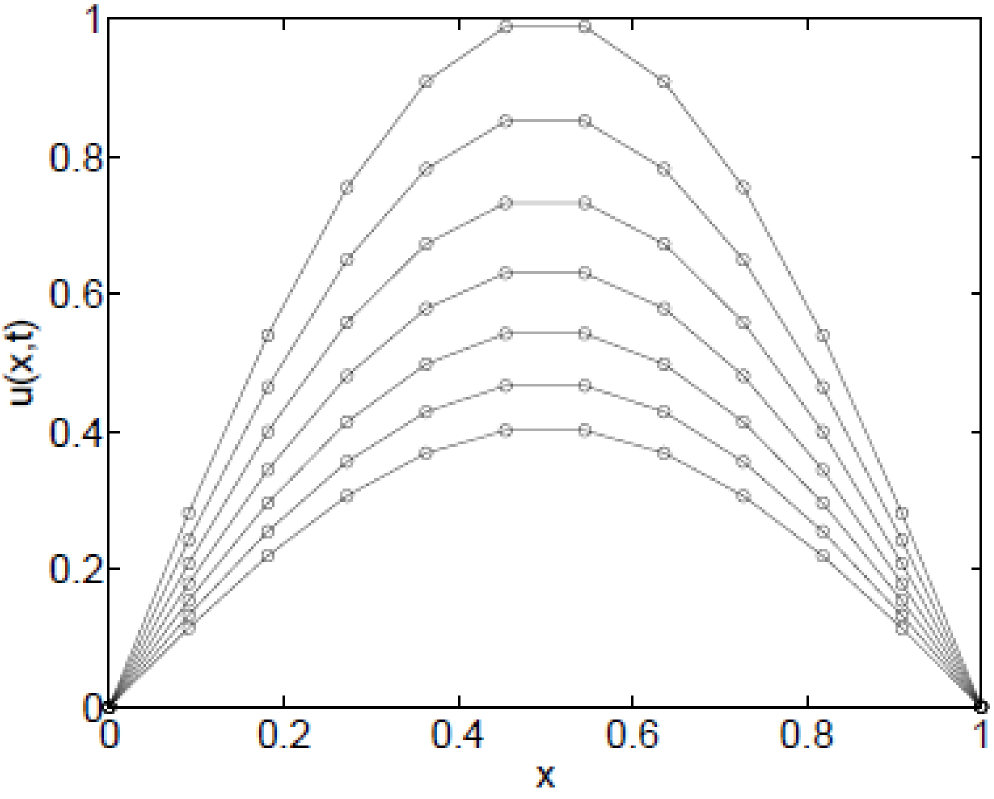
Then it has a unique solution $u(x,t) = \sin(\pi x)e^{-\pi t}$. (5)

```
%%%%%%%%%%%%%%%%%%%%%%%%%%% heatim.m %%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear; clc; clf; Nx=12; x=linspace(0,1,Nx); h=x(2)-x(1);
T=0.1; alpha=2; k=alpha*(h^2); Nt=round(T/k);
u(:,1)=sin(pi*x);
for i=1:Nx-2
    dd(i)= 1 + 2*alpha; c(i)= - alpha; a(i)= - alpha;
end
for n=1:Nt
    d=dd;
    for i=1:Nx-2
        b(i)=u(i+1,n);
    end
    for i=2:Nx-2
    xmult= a(i-1)/d(i-1);
    d(i) = d(i) - xmult*c(i-1);
    b(i) = b(i) - xmult*b(i-1);
    end
       u(Nx-1,n+1) = b(Nx-2)/d(Nx-2);
    for i = Nx-3:-1:1
       u(i+1,n+1) = (b(i) - c(i)*u(i+2,n+1))/d(i);
    end
end
plot(x,u,'ko-')
xlabel('x','FontSize',20); ylabel('u(x,t)','FontSize',20);
```

$$d_i = d_i - \frac{a_{i-1}}{d_{i-1}}c_{i-1}, \qquad b_i = b_i - \frac{a_{i-1}}{d_{i-1}}b_{i-1} \qquad (1)$$

$$x_{N_x} = \frac{b_{N_x}}{d_{N_x}},$$

$$x_i = \frac{1}{d_i}(b_i - c_i x_{i+1}), \qquad (2)$$



u(x,t)
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
x

Crank-Nicolson method

This method for the heat equation has no stability condition and is second order in both space and time. This scheme is called the Crank-Nicolson method and is one of the most popular methods in practice.

We write the equation at the point $(x_i, t^{n+1/2})$. Then

$$u_t(x_i, t^{n+1/2}) = \frac{u_i^{n+1} - u_i^n}{k} + O(k^2) \qquad (1)$$

is a centered difference approximation for u_t at $(x_i, t^{n+1/2})$ and therefore should be $O(\triangle t^2)$.

To approximate the term $u_{xx}(x_i, t^{n+1/2})$, we use the average of the second centered difference for $u_{xx}(x_i, t^{n+1})$ and $u_{xx}(x_i, t^n)$. That is

$$\begin{aligned} u_{xx}(x_i, t^{n+1/2}) &= \frac{1}{2}\left(u_{xx}(x_i, t^n) + u_{xx}(x_i, t^{n+1})\right) + O(h^2) \\ &= \frac{1}{2}\left(\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} + \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2}\right) \\ &\quad + O(h^2). \end{aligned} \qquad (2)$$

We let $\alpha = \frac{k}{h^2}$ and rewrite the equation such as

$$-\alpha u_{i-1}^{n+1} + 2(1+\alpha)u_i^{n+1} - \alpha u_{i+1}^{n+1} = \alpha u_{i-1}^n + 2(1-\alpha)u_i^n + \alpha u_{i+1}^n. \quad (1)$$

The one must solve a system of linear equations.

From (1), we solve the system

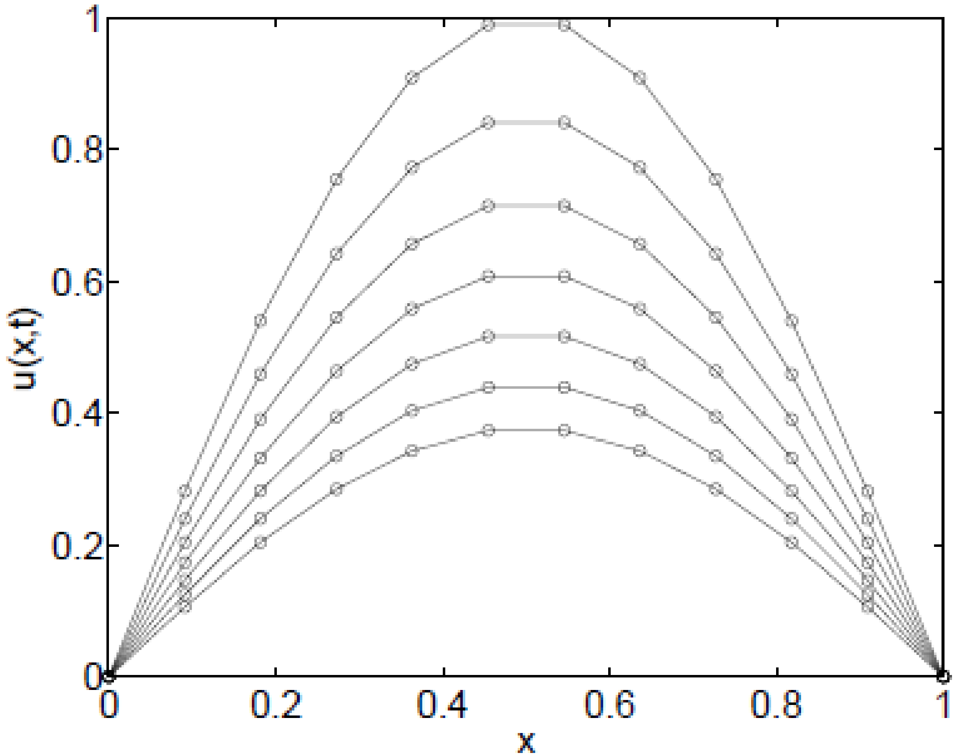
$$\begin{pmatrix} 2(1+\alpha) & -\alpha & & & & \\ -\alpha & 2(1+\alpha) & -\alpha & & & \\ & -\alpha & 2(1+\alpha) & -\alpha & & \\ & \ddots & \ddots & \ddots & & \\ & & & & 2(1+\alpha) & -\alpha \\ & & & & -\alpha & 2(1+\alpha) \end{pmatrix} \begin{pmatrix} u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \\ \vdots \\ u_{Nx-2}^{n+1} \\ u_{Nx-1}^{n+1} \end{pmatrix} = \begin{pmatrix} \alpha u_1^{n+1} + 2(1-\alpha)u_2^n + \alpha u_3^n \\ \alpha u_2^n + 2(1-\alpha)u_3^n + \alpha u_4^n \\ \alpha u_3^n + 2(1-\alpha)u_4^n + \alpha u_5^n \\ \vdots \\ 2(1-\alpha)u_{Nx-2}^n + \alpha u_{Nx-1}^n \\ \alpha u_{Nx-2}^n + 2(1-\alpha)u_{Nx-1}^n + \alpha u_{Nx}^{n+1} \end{pmatrix} = \begin{pmatrix} b_2^n \\ b_3^n \\ b_4^n \\ \vdots \\ b_{Nx-2}^n \\ b_{Nx-1}^n \end{pmatrix}$$

The system can be written symbolically as

$$\mathbf{A}\mathbf{u}^{n+1} = \mathbf{b}^n. \quad (2)$$

```
%%%%%%%%%%%%%%%%%%%%%%%%%%% heatCN.m %%%%%%%%%%%%%%%%%%%%%%%%%%%
clear; clc; clf; Nx=12; x=linspace(0,1,Nx); h=x(2)-x(1);
T=0.1; alpha = 2; k = alpha*(h^2); Nt=round(T/k);
u(:,1)=sin(pi*x);
for i=1:Nx-2
    dd(i)= 2*(1+alpha); c(i)= - alpha; a(i)= - alpha;
end
for n=1:Nt
    d=dd;
    for i=1:Nx-2
      b(i)=alpha*u(i,n)+2*(1-alpha)*u(i+1,n)+alpha*u(i+2,n);
    end
    for i = 2:Nx-2
        xmult=a(i-1)/d(i-1);
        d(i)=d(i)-xmult*c(i-1);  b(i)=b(i)-xmult*b(i-1);
    end
    u(Nx-1,n+1) = b(Nx-2)/d(Nx-2);
    for i = Nx-3:-1:1
        u(i+1,n+1) = (b(i) - c(i)*u(i+2,n+1))/d(i);
    end
end
plot(x,u,'ko-');
xlabel('x','FontSize',20); ylabel('u(x,t)','FontSize',20)
```

$$-\alpha u_{i-1}^{n+1} + 2(1+\alpha)u_i^{n+1} - \alpha u_{i+1}^{n+1} = \alpha u_{i-1}^n + 2(1-\alpha)u_i^n + \alpha u_{i+1}^n \qquad (1)$$



u(x,t)
x
0
0.2
0.4
0.6
0.8
1

Stability of Crank-Nicolson

In this part we show the stability of Crank-Nicolon using von Neumann analysis. Let $u_k^n = e^{i\beta kh}\xi^n$ and apply this for (1).

$$-\alpha u_{i-1}^{n+1} + 2(1+\alpha)u_i^{n+1} - \alpha u_{i+1}^{n+1} = \alpha u_{i-1}^n + 2(1-\alpha)u_i^n + \alpha u_{i+1}^n \quad (1)$$

Then we obtain

$$\begin{aligned} &-\alpha e^{i\beta(k-1)h}\xi^{n+1} + (1+2\alpha)e^{i\beta kh}\xi^{n+1} - \alpha e^{i\beta(k+1)h}\xi^{n+1} \\ &\qquad = \alpha e^{i\beta(k-1)h}\xi^n + 2(1-\alpha)e^{i\beta kh}\xi^n + \alpha e^{i\beta(k+1)h}\xi^n, \quad (2) \\ &-\alpha e^{-i\beta h}\xi + (1+2\alpha)\xi - \alpha e^{i\beta h}\xi = \alpha e^{i\beta h} + 2(1-\alpha) + \alpha e^{i\beta h}, \\ &2(1+\alpha(1-\cos(\beta h)))\xi = 2(1-\alpha(1-\cos(\beta h)). \quad (3) \end{aligned}$$

Therefore,

$$\xi = \frac{1-2\alpha\sin^2(\beta h/2)}{1+2\alpha\sin^2(\beta h/2)}. \quad (4)$$

This gives us that $\frac{1-2\alpha}{1+2\alpha} \leq \xi \leq 1$. (5)

Prevention of spurious oscillations

We have observed the Crank-Nicolson scheme's advantages. However, applying it to the below problem reveals a subtle weakness.

See the heat equation $\partial_t u(x,t) = u_{xx}(x,t),\ 0 < x < 1,\ t > 0.$ (1)

But we have different initial conditions

$$u(x,0) = 0.3H(x) + 0.4H(x-0.5) \tag{2}$$

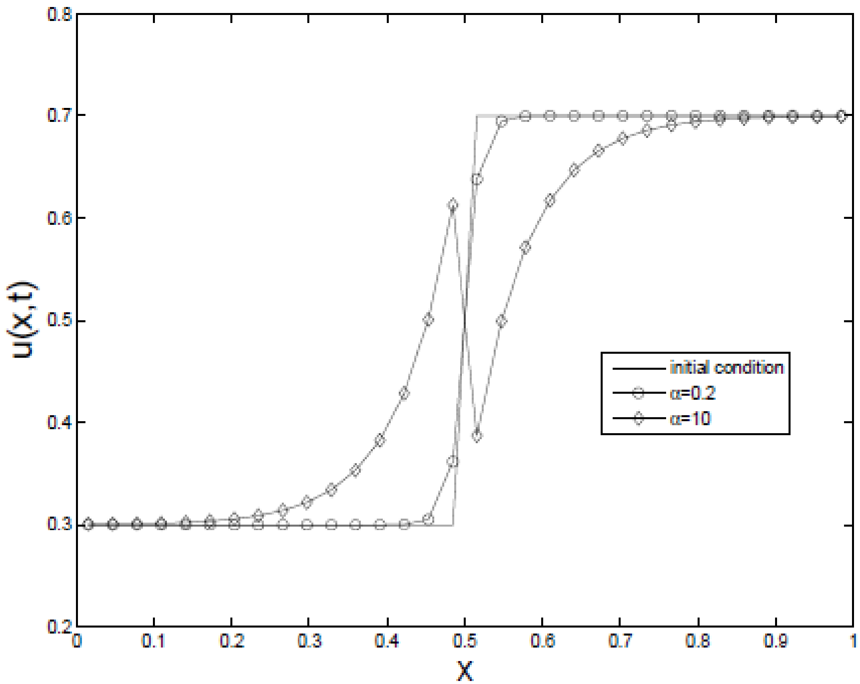
where $H(x)$ denotes the Heaviside step function.

Then, using the Crank-Nicolson method,

$$\begin{pmatrix} (2+\alpha) & -\alpha & & & & \\ -\alpha & 2(1+\alpha) & -\alpha & & & \\ & -\alpha & 2(1+\alpha) & -\alpha & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\alpha & 2(1+\alpha) & -\alpha \\ & & & & -\alpha & (2+\alpha) \end{pmatrix} \begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ \vdots \\ u_{Nx-1}^{n+1} \\ u_{Nx}^{n+1} \end{pmatrix} = \begin{pmatrix} (2-\alpha)u_1^n + \alpha u_2^n \\ \alpha u_1^n + 2(1-\alpha)u_2^n + \alpha u_3^n \\ \alpha u_2^n + 2(1-\alpha)u_3^n + \alpha u_4^n \\ \vdots \\ \alpha u_{Nx-2}^n + 2(1-\alpha)u_{Nx-1}^n + \alpha u_{Nx}^n \\ \alpha u_{Nx-1}^n + (2-\alpha)u_{Nx}^n \end{pmatrix} = \begin{pmatrix} b_1^n \\ b_2^n \\ b_3^n \\ \vdots \\ b_{Nx-1}^n \\ b_{Nx}^n \end{pmatrix}. \tag{3}$$

```
%%%%%%%%%%%%%%%%%%%%%%%%%%% osc_heatCN.m %%%%%%%%%%%%%%%%%%%%%%%%
clear; clc; clf; Nx=32; h=1/Nx; x=linspace(0.5*h,1-0.5*h,Nx);
for i=1:Nx
    if (x(i)<0.5)
        u(i,1)=0.3;
    else
        u(i,1)=0.7;
    end
end
plot(x,u(:,1),'k-'); hold
for iter=1:2
alpha = 0.2+9.8*(iter-1);
k = alpha*h^2;
for i=1:Nx
    dd(i)= 2*(1+alpha); c(i)= -alpha; a(i)= -alpha;
end
dd(1)=2+alpha; dd(Nx)=2+alpha;
for n=1:1
    d=dd;
    for i=2:Nx-1
      b(i)=alpha*u(i-1,n)+2*(1-alpha)*u(i,n)+alpha*u(i+1,n);
    end
    b(1)=(2-alpha)*u(1,n)+alpha*u(2,n);
    b(Nx)=alpha*u(Nx-1,n)+(2-alpha)*u(Nx,n);
    for i = 1:Nx-1
        xmult=a(i)/d(i);
        d(i+1)=d(i+1)-xmult*c(i);  b(i+1)=b(i+1)-xmult*b(i);
    end
    u(Nx,n+1) = b(Nx)/d(Nx);
```

```
    for i = Nx-1:-1:1
        u(i,n+1) = (b(i) - c(i)*u(i+1,n+1))/d(i);
    end
end
if iter==1
plot(x,u(:,2),'ko-');
else
plot(x,u(:,2),'kd-');
end
end
axis([0 1 0.2 0.8])
legend('initial condition','\alpha=0.2','\alpha=10')
xlabel('x','FontSize',20); ylabel('u(x,t)','FontSize',20)
```



To prevent spurious oscillations, we analyze the methods such as explicit, implicit, and Crank-Nicolson schemes. Let us suppose that we have the solution in infinite domain to avoid boundary conditions. Then the heat equation is rewritten as

$$\begin{aligned}(1+2\theta\alpha)u_i^{n+1} &= (1-\theta)\alpha u_{i-1}^n + (1-2(1-\theta)\alpha)u_i^n + (1-\theta)\alpha u_{i+1}^n \\ &\quad +\theta\alpha u_{i-1}^{n+1} + \theta\alpha u_{i+1}^{n+1},\end{aligned} \quad (1)$$

where $\alpha = k/h^2$.

We can get explicit, implicit, and Crank-Nicolson when $\theta = 0,\ \theta = 1,\ \theta = 1/2,$ respectively.
Next, we define $u_i^{\max} = \max\{u_{i-1}^n, u_i^n, u_{i+1}^n, u_{i-1}^{n+1}, u_{i+1}^{n+1}\}$
and apply the maximum principle, so that

$$\begin{aligned}(1+2\theta\alpha)u_i^{n+1} &\leq (1-\theta)\alpha u_i^{\max} + (1-2(1-\theta)\alpha)u_i^{\max} + (1-\theta)\alpha u_i^{\max} \\ &\quad +\theta\alpha u_i^{\max} + \theta\alpha u_i^{\max} = (1+2\theta\alpha)u_i^{\max}.\end{aligned} \quad (2)$$

It follows $u_i^{n+1} \leq u_i^{\max}$ and (3)
similarly, define $u_i^{\min} = \min\{u_{i-1}^n, u_i^n, u_{i+1}^n, u_{i-1}^{n+1}, u_{i+1}^{n+1}\}$ and then apply the minimum principle for (1)

$$u_i^{n+1} \geq u_i^{\min}. \quad (4)$$