For a state for $l=0$, $\psi(x) \to \psi^*(x)$

$$\Theta|\alpha\rangle = \Theta\int dx\,|x\rangle\langle x|\alpha\rangle = \int dx\,|x\rangle\langle x|\alpha\rangle^*.$$

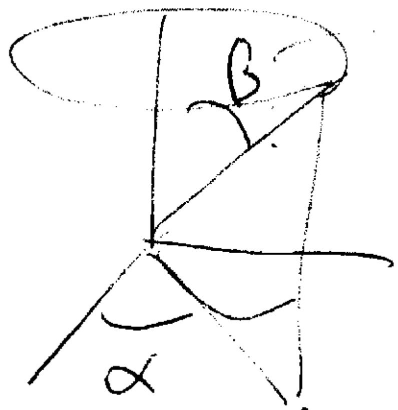
if we expand in terms of the momentum eigenstate,

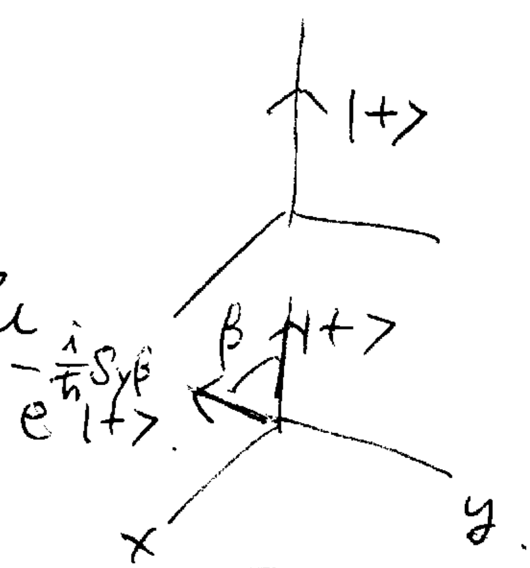
$$\Theta|\alpha\rangle = \Theta\int dp\,|p\rangle\langle p|\alpha\rangle = \int dp\,|-p\rangle\langle p|\alpha\rangle^*.$$

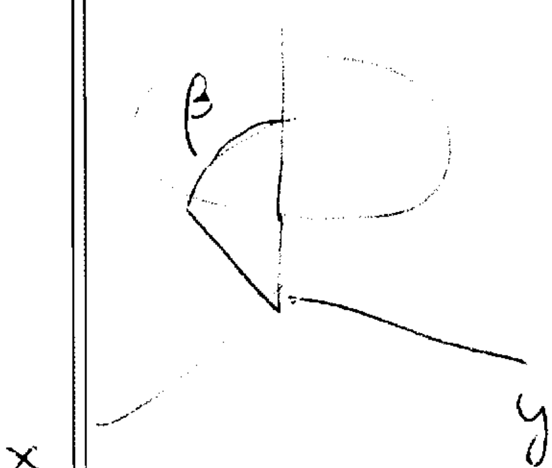
$$\underline{\underline{\phi(p) \to \phi^*(-p)}}$$

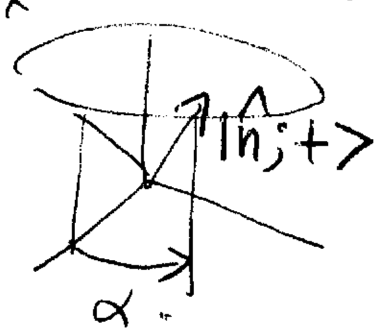
Time reversal for a spin $\frac{1}{2}$ system

Sec. 3.2



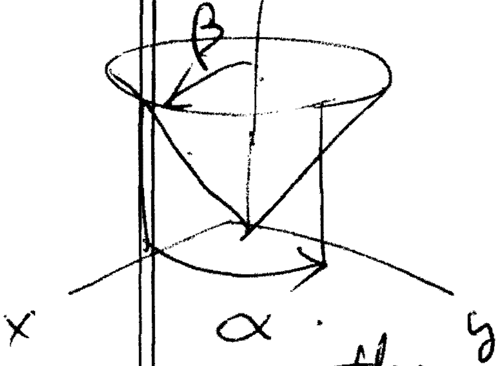






$$e^{-\frac{i}{\hbar}S_z\alpha}\, e^{-\frac{i}{\hbar}S_y\beta}|+\rangle$$

$$\therefore |\hat{n};+\rangle = \underbrace{e^{-\frac{i}{\hbar}S_z\phi}\, e^{-\frac{i}{\hbar}S_y\beta}\, \underbrace{|+\rangle}_{\uparrow \text{ along } \hat{z}}}_{\uparrow \text{ along } \hat{n}}$$



applying the time-reveral operator Θ,

$$\Theta |\hat{n};+\rangle = e^{-\frac{i}{\hbar}S_z\alpha}\, e^{-\frac{i}{\hbar}S_y\beta}\, \Theta|+\rangle$$

note that. Θ

$$\Theta S_z \Theta^{-1} = -S_z \quad\rightarrow\quad \Theta e^{-\frac{i}{\hbar}S_z\phi}\Theta^{-1} = e^{-\frac{i}{\hbar}S_z\phi}$$

$$\Theta S_y \Theta^{-1} = -S_y \quad\rightarrow\quad \Theta e^{-\frac{i}{\hbar}S_y\beta}\Theta^{-1} = e^{-\frac{i}{\hbar}S_y\beta}$$

☆ $\underline{\Theta\, i\, \Theta^{-1} = -i}$

$$\Theta|\hat{n};+\rangle = \eta|\hat{n};-\rangle$$

rotat $e^{-\frac{i}{\hbar}S_y\pi}K|+\rangle$

$$|\hat{n};-\rangle = e^{-\frac{i}{\hbar}S_z\alpha}\, e^{-\frac{i}{\hbar}S_y(\pi+\beta)}|+\rangle$$

" Because $K|+\rangle = |+\rangle$

$$\eta|\hat{n};-\rangle = \eta\, e^{-\frac{i}{\hbar}S_z\alpha}\, e^{-\frac{i}{\hbar}S_y\beta}\, \underbrace{e^{-\frac{i}{\hbar}S_y\pi}|+\rangle}$$

$$= e^{-\frac{i}{\hbar}S_z\alpha}\, e^{-\frac{i}{\hbar}S_y\beta}\, \underbrace{\Theta|+\rangle}$$

$$\therefore \Theta = \eta\, \underbrace{e^{-\frac{i}{\hbar}S_y\pi}}_{\text{unitary part } U}\, K$$

K ← complex conjugate op.

$$\Theta = \eta\, e^{-\frac{i}{\hbar}S_y\pi}$$

$$e^{-\frac{i}{\hbar}S_y\pi} = e^{-\frac{i}{\hbar}\frac{\hbar}{2}\sigma_2\pi} = e^{-\frac{i}{2}\pi\sigma_2}$$

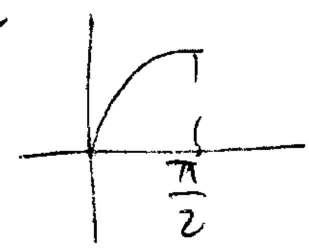
$$e^{i\vec{\sigma}\cdot\vec{a}} = \sum_{k=0}^{\infty} i^{2k}\frac{(\vec{\sigma}\cdot\vec{a})^{2k}}{(2k)!} + \sum_{k=0}^{\infty} i\frac{(\vec{\sigma}\cdot\vec{a})^{2k+1}\, i^{2k}}{(2k+1)!}$$

$$= \mathbb{1}\sum_{k=0}^{\infty}(-1)^k\frac{a^2}{(2k)!} + i\,\vec{\sigma}\cdot\hat{a}\sum_{k=0}^{\infty}\frac{a^{2k+1}}{(2k+1)!}(-1)^k$$

$$(\vec{\sigma}\cdot\vec{a})^2 = \sigma^i\sigma^j a^i a^j = \frac{1}{2}(\sigma^i\sigma^j + \sigma^j\sigma^i)a^i a^j$$

$$= \delta^{ij} a^i a^j = a^2$$

$$= \mathbb{1}\cos a + i\,\vec{\sigma}\cdot\hat{a}\sin a$$



$$e^{-\frac{i}{\hbar}S_y\pi} = e^{-\frac{i}{2}\pi\sigma_2}$$

$$= \mathbb{1}\cos\left(-\frac{\pi}{2}\right) + i\,\sigma_2 \sin\left(-\frac{\pi}{2}\right)$$

$$\sin\left(-\frac{\pi}{2}\right) = -1$$

$$= -i\sigma_2 \qquad \sigma_2 = \frac{2}{\hbar}S_y \qquad \cos\left(-\frac{\pi}{2}\right) = 0$$

$$\therefore\ \Theta = \eta\, e^{-\frac{i}{\hbar}S_y\pi}K = -\eta\, i\,\sigma_2 K = -i\eta\left(\frac{2S_y}{\hbar}\right)K$$

η: arbitrary phase

$$S_y = \frac{\hbar}{2}\sigma_2$$

$$\Theta = -i\eta\left(\frac{2S_y}{\hbar}\right)K = -i\eta\sigma_2 K$$

$$|+\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$e^{i\vec{\sigma}\cdot\vec{a}} = \mathbb{1}\cos a + i\vec{\sigma}\cdot\hat{a}\sin a$$

$$\sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix} \qquad \sigma_2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix} \qquad \sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\sigma_3\sigma_1 = \begin{pmatrix}1&\\&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&1\\-1&0\end{pmatrix} = i\begin{pmatrix}0&-i\\i&0\end{pmatrix} = i\sigma_2$$

$$\begin{aligned}|\hat{n};+\rangle &= e^{-\frac{i}{\hbar}S_z\phi}\, e^{-\frac{i}{\hbar}S_y\theta} \\ &= e^{-\frac{i}{\hbar}\frac{\hbar}{2}\sigma_3\phi}\, e^{-\frac{i}{\hbar}\frac{\hbar}{2}\sigma_2\theta} \\ &= e^{+i\sigma_3(-\frac{\phi}{2})}\, e^{+i\sigma_2(-\frac{\theta}{2})}\end{aligned}$$

$$\begin{aligned}e^{i\sigma_3(-\frac{\phi}{2})} &= \mathbb{1}\cos\left(-\frac{\phi}{2}\right) + i\sigma_3\sin\left(-\frac{\phi}{2}\right) \\ &= \mathbb{1}\cos\frac{\phi}{2} + \begin{pmatrix}i&0\\0&-i\end{pmatrix}\left(-\sin\frac{\phi}{2}\right) \\ &= \begin{pmatrix}\cos\frac{\phi}{2} - i\sin\frac{\phi}{2} & 0 \\ 0 & \cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\end{pmatrix} = \begin{pmatrix}e^{-\frac{i}{2}\phi} & 0 \\ 0 & e^{\frac{i}{2}\phi}\end{pmatrix}\end{aligned}$$

$$e^{i\sigma_2(-\frac{\theta}{2})} = \mathbb{1}\cos\left(-\frac{\theta}{2}\right) + i\sigma_2\sin\left(-\frac{\theta}{2}\right) \qquad -i\sigma_2 = -i\begin{pmatrix}&-i\\i&\end{pmatrix} = \begin{pmatrix}&-1\\1&\end{pmatrix}$$

$$= \begin{pmatrix}\cos\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2}\end{pmatrix} + \begin{pmatrix}0 & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & 0\end{pmatrix}$$

$$= \begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}$$

$$\therefore\ e^{i\sigma_3\left(-\frac{\phi}{2}\right)}\, e^{i\sigma_2\left(-\frac{\theta}{2}\right)}$$

$$= \begin{pmatrix} e^{-\frac{i}{2}\phi} & 0 \\ 0 & e^{\frac{i}{2}\phi} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-\frac{i}{2}\phi}\cos\frac{\theta}{2} & -e^{-\frac{i}{2}\phi}\sin\frac{\theta}{2} \\ e^{\frac{i}{2}\phi}\sin\frac{\theta}{2} & e^{\frac{i}{2}\phi}\cos\frac{\theta}{2} \end{pmatrix}$$

$$\therefore\ |\hat{n};+\rangle = \begin{pmatrix} & " & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-\frac{i}{2}\phi}\cos\frac{\theta}{2} \\ e^{\frac{i}{2}\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

$$|\hat{n};-\rangle = \begin{pmatrix} & & \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -e^{-\frac{i}{2}\phi}\sin\frac{\theta}{2} \\ e^{\frac{i}{2}\phi}\cos\frac{\theta}{2} \end{pmatrix}$$

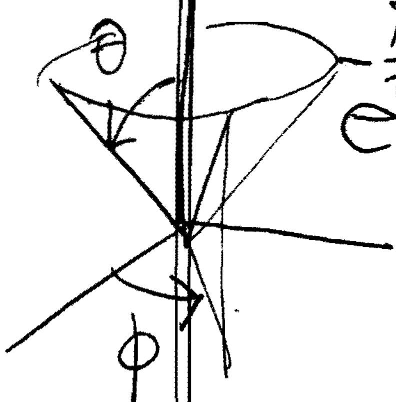
$$\Theta|\hat{n};+\rangle = -i\eta\sigma_2 K\,|\hat{n};+\rangle \qquad -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

same!

$$= \eta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} K \begin{pmatrix} e^{-\frac{i}{2}\phi}\cos\frac{\theta}{2} \\ e^{\frac{i}{2}\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

complex conjugate

$$= \eta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{\frac{i}{2}\phi}\cos\frac{\theta}{2} \\ e^{-\frac{i}{2}\phi}\sin\frac{\theta}{2} \end{pmatrix} = \eta \begin{pmatrix} -e^{-\frac{i}{2}\phi}\sin\frac{\theta}{2} \\ e^{\frac{i}{2}\phi}\cos\frac{\theta}{2} \end{pmatrix}$$



$$e^{-\frac{i}{\hbar}S_z\phi}\, e^{-\frac{i}{\hbar}S_y\theta}.$$

$$|n;+\rangle = e^{-\frac{i}{\hbar}S_z\phi}\, e^{-\frac{i}{\hbar}S_y\theta}|+\rangle.$$

$$|n;-\rangle = e^{-\frac{i}{\hbar}S_z\phi}\, e^{-\frac{i}{\hbar}S_y(\theta+\pi)}|+\rangle$$

$$\underset{|+\rangle}{\begin{pmatrix}1\\0\end{pmatrix}} \longrightarrow . \underset{|-\rangle.}{\begin{pmatrix}0\\1\end{pmatrix}}$$

$$e^{-\frac{i}{\hbar}S_y\pi} = e^{-\frac{i}{\hbar}\frac{\hbar}{2}\sigma_2\pi} = e^{i\sigma_2(-\frac{\pi}{2})} \qquad i\sigma_2 = i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

$$= \mathbb{1}\cos\left(-\frac{\pi}{2}\right) + i\sigma_2\sin\left(-\frac{\pi}{2}\right) \qquad = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\cos\frac{\pi}{2} + \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\sin\frac{\pi}{2} \qquad \cos\frac{\pi}{2}=0,\ \sin\frac{\pi}{2}=1$$

$$= \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

$$\therefore \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}.$$

$$H|\hat{n};+\rangle = \eta|\hat{n};-\rangle$$

$$H e^{-\frac{i}{\hbar}S_z\phi}\, e^{-\frac{i}{\hbar}S_y\theta}|+\rangle = \eta\, e^{-\frac{i}{\hbar}S_z\phi}\, e^{-\frac{i}{\hbar}S_y(\theta+\pi)}|+\rangle.$$

$$H e^{-\frac{i}{\hbar}S_z\phi} H^{-1} = e^{-\frac{(-i)}{\hbar}(-S_z)\phi} = e^{-\frac{i}{\hbar}S_z\phi}$$

$$\therefore H|+\rangle = \eta\, e^{-\frac{i}{\hbar}S_y\pi}|+\rangle = \eta\, e^{-\frac{i}{\hbar}S_y\pi}K|+\rangle$$

Because $K|+\rangle = |+\rangle$ ($S_z = \frac{\hbar}{2}\begin{pmatrix}1 & \\ & -1\end{pmatrix}$ real & $|+\rangle \begin{pmatrix}1\\0\end{pmatrix}, |-\rangle\begin{pmatrix}0\\1\end{pmatrix}$.

$$\Theta = \eta e^{-\frac{i}{\hbar} S_y \pi} K = \eta e^{-\frac{i}{\hbar}\frac{\hbar}{2}\sigma_2 \pi} K$$

$$i\sigma_2 = i\begin{pmatrix} & -i \\ i & \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \eta e^{i\sigma_2\left(-\frac{\pi}{2}\right)} K$$

$$= \eta \left[\cancel{\mathbb{1}\cos\left(-\frac{\pi}{2}\right)} + i\sigma_2 \sin\left(-\frac{\pi}{2}\right)\right] K$$

$$= \eta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} K \qquad \sin\frac{\pi}{2} = 1.$$

$$\Theta|+\rangle = \eta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} K \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \eta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \eta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Theta|-\rangle = \eta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} K \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \eta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\eta \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\therefore \left.\begin{aligned} \Theta|+\rangle &= \eta|-\rangle \\ \Theta|-\rangle &= -\eta|+\rangle \end{aligned}\right\} \Rightarrow \Theta|\pm\rangle = \pm\eta|\mp\rangle$$

$$\therefore \Theta\left[c_+|+\rangle + c_-|-\rangle\right]$$

$$= \eta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} K \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \eta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_+^* \\ c_-^* \end{pmatrix} = \eta \begin{pmatrix} -c_-^* \\ c_+^* \end{pmatrix}$$

$$= \eta\left[-c_-^*|+\rangle + c_+^*|-\rangle\right]$$

$$\Theta^2 = \eta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} K \, \eta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} K \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

$$=$$

$$\Theta^2 (c^+|+\rangle + c^-|-\rangle) = \eta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} K\eta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} K \begin{pmatrix} c+ \\ c- \end{pmatrix}$$

$(z^*)^* = z$

$$\boxed{K\eta = \eta^* K}, \quad K^2 = 1.$$

$$= \eta\eta^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} K^2 \begin{pmatrix} c+ \\ c- \end{pmatrix}$$

$$= |\eta|^2 \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c+ \\ c- \end{pmatrix}$$

$|\eta|^2 = 1$ unit modulous.

$$= -\begin{pmatrix} c+ \\ c- \end{pmatrix}$$

$$= -(c^+|+\rangle + c^-|-\rangle).$$

$$\therefore \boxed{\Theta^2 = -\mathbb{1}}$$ independent of the phase η.

spin-$\frac{1}{2}$ state

$\Theta^2 = +\mathbb{1}$ for spin 0 state

We now prove that

$$\Theta^2 |j = \text{half-integer}\rangle = -|j\rangle$$

$$\Theta^2 |j = \text{integer}\rangle = +|j\rangle$$

Therefore, eigenvalue of Θ^2 is $(-1)^{2j}$.

We recall that, for a spin-$\frac{1}{2}$ system,

$$\Theta = \eta e^{-\frac{i}{\hbar}S_y\pi}K$$

In general,

$$\begin{cases} |\hat{n};jm\rangle = e^{-\frac{i}{\hbar}J_z\phi} e^{-\frac{i}{\hbar}J_y\theta} |\hat{z};jm\rangle \\ \Theta|\hat{n};jm\rangle = \Theta e^{-\frac{i}{\hbar}J_z\phi} e^{-\frac{i}{\hbar}J_y\theta} |\hat{z};jm\rangle \end{cases}$$

$$\Theta i \Theta^{-1} = -i$$

$$\Theta \vec{J} \Theta^{-1} = -\vec{J}$$

$$\Rightarrow \Theta e^{-\frac{i}{\hbar}\vec{J}\cdot\vec{\phi}} \Theta^{-1} = e^{-\frac{i}{\hbar}\vec{J}\cdot\vec{\phi}}$$

$$= e^{-\frac{i}{\hbar}J_z\phi} e^{-\frac{i}{\hbar}J_y\theta} \underbrace{\Theta|\hat{z};jm\rangle}$$

$$= \quad '' \quad \eta|\hat{z};j-m\rangle$$

$$= \eta\left[e^{-\frac{i}{\hbar}J_z\phi} e^{-\frac{i}{\hbar}J_y\theta}|\hat{z};j-m\rangle\right]$$

$$= \eta|\hat{n};j-m\rangle$$

On the other hand

$$|\hat{n};j-m\rangle = e^{-\frac{i}{\hbar}J_z\phi} e^{-\frac{i}{\hbar}J_y\theta} e^{-\frac{i}{\hbar}J_y\pi}|\hat{z};j+m\rangle$$

$$= e^{-\frac{i}{\hbar}J_z\phi} e^{-\frac{i}{\hbar}J_y\theta} \eta e^{-\frac{i}{\hbar}J_y\pi}|\hat{z};j,+m\rangle$$

$\therefore \Theta = \eta e^{-\frac{i}{\hbar}J_y\pi}\underbrace{K}_{\text{complex conjugation operator}}$ for a J system with

$$\boxed{\Theta = \eta\, e^{-\frac{i}{\hbar}J_y\pi} K}$$

$$|\alpha\rangle = \sum |jm\rangle\langle jm|\alpha\rangle$$

Note that $J_y|jm\rangle \neq m\hbar|jm\rangle$ not

but $J_z|jm\rangle = m\hbar|jm\rangle$

~~Applying~~ Θ^2

$$\Theta^2 = \eta\, e^{-\frac{i}{\hbar}J_y\pi} K\ \eta\, e^{-\frac{i}{\hbar}J_y\pi} K$$

$$= \eta\, e^{-\frac{i}{\hbar}J_y\pi}\eta^* K\, e^{-\frac{i}{\hbar}J_y\pi} K$$

$$= |\eta|^2 e^{-\frac{i}{\hbar}J_y\pi}\, e^{+\frac{i}{\hbar}J_y^*\pi} K^2$$

$$e^{-\frac{i}{\hbar}J_y(2\pi)}$$

$$\|$$

$$K^2 = \mathbb{1}$$

$$= |\eta|^2 e^{-\frac{i}{\hbar}J_y\pi}\, e^{+\frac{i}{\hbar}J_y^*\pi} = |\eta|^2 e^{-\frac{i}{\hbar}J_y(2\pi)}$$

$$J_\pm = J_x \pm iJ_y \qquad J_y^* = -J_y$$

$$\therefore\ J_y = \frac{1}{2i}(J_+ - J_-) \text{ and } J_x = \frac{1}{2}(J_+ + J_-)$$

$$\langle jm'|J_\pm|jm\rangle = \hbar\sqrt{j(j+1) - m(m\pm1)}\ \delta_{m'\,m\pm1}$$

Therefore, $J_\pm$ are real.

$\Rightarrow$ J_y is pure imaginary and J_x is real for any J.

Therefore, $J_y^* = -J_y$

Therefore, $\Theta^2 = e^{-\frac{i}{\hbar}J_y(2\pi)}$

This is the rotation about the y axis by an angle of 2π.

Homework)

Show that $e^{-\frac{i}{\hbar}J_y(2\pi)}|jm\rangle = (-1)^{2j}|jm\rangle$

One example is $J_y = \frac{\hbar}{2}\sigma_2$

$$e^{-\frac{i}{\hbar}J_y(2\pi)} = e^{-\frac{i}{\hbar}\frac{\hbar}{2}\sigma_2(2\pi)} = e^{i\sigma_2(-\pi)}$$

$$= \mathbb{1}\cos(-\pi) + i\sigma_2\sin(-\pi)$$

$$= \cos\pi\,\mathbb{1} = -\mathbb{1}.$$