KECE321 Communication Systems I

(Haykin Sec. 5.1 - Sec. 5.2)

Lecture #20, May 21, 2012

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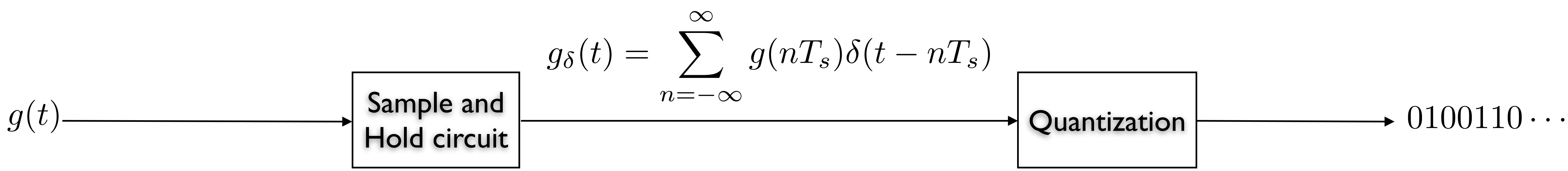
Summary

- Sampling theorem
- Pulse-amplitude modulation (PAM)

Pulse Modulation: Transition from Analog to Digital Communications

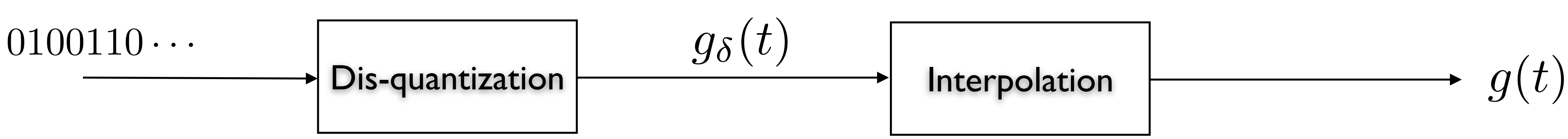
- Sampling process
- Pulse-amplitude modulation
- Pulse-position modulation
- Quantization
- Pulse code modulation
- Delta modulation

From Analog to Digital

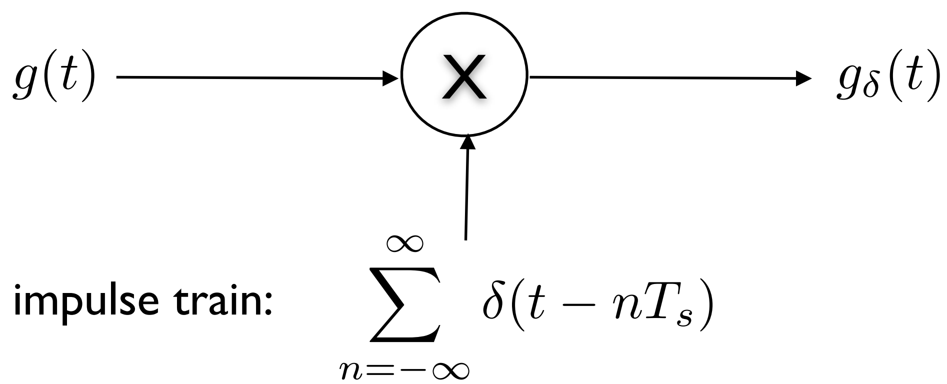


$$T_s \quad : \quad \text{sampling period}$$

$$f_s = \frac{1}{T_s} : \text{sampling rate}$$



Sampling Process



$$
\begin{aligned}
g_\delta(t) &= g(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \\
&= \sum_{n=-\infty}^{\infty} g(t)\delta(t - nT_s) \\
&= \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s)
\end{aligned}
$$

- Two questions
  - What are the restriction on $g(t)$ and T_s to allow perfect recovery of $g(t)$ from $g_\delta(t)$?
  - How is $g(t)$ recovered from $g_\delta(t)$?

- Fourier transform of $g_\delta(t)$

$$g_\delta(t) = g(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \quad \longleftrightarrow \quad G_\delta(f) = G(f) * \mathcal{F}\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right]$$

- Now let us find $\mathcal{F}\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right]$.

  - First of all, we note that the impulse train is periodic signal with a fundamental period T_s. Hence, it can be described in Fourier series form such as

$$\sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_s t}$$

where the Fourier coefficient c_n can be found as

$$c_n = \frac{1}{T_s} \int_{T_s} \delta(t) e^{-j2\pi n f_s t}\, dt = f_s$$

  - Hence, we have $$\sum_{n=-\infty}^{\infty} \delta(t - nT_s) = f_s \sum_{n=-\infty}^{\infty} e^{j2\pi n f_s t}$$

- Now taking the Fourier transform of the impulse train in Fourier series form yields

$$
\begin{aligned}
\mathcal{F}\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right] &= \int_{-\infty}^{\infty} \left(f_s \sum_{n=-\infty}^{\infty} e^{-j2\pi n f_s t} \right) e^{-j2\pi f t}\, dt \\
&= f_s \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j2\pi (f - n f_s) t}\, dt \\
&= f_s \sum_{n=-\infty}^{\infty} \delta(f - n f_s)
\end{aligned}
$$

- where we make use of

$$\delta(t) \longleftrightarrow 1 \qquad \text{and} \qquad 1 \longleftrightarrow \delta(f)$$

$$\Longrightarrow \int_{-\infty}^{\infty} e^{-j2\pi f t}\, dt = \delta(f)$$

- Hence, the Fourier transform of $g_\delta(t)$ can be written as

$$\begin{aligned} G_\delta(f) &= G(f) * \left[f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \right] \\ &= f_s \sum_{n=-\infty}^{\infty} G(f - nf_s) \end{aligned}$$

- We also find $G_\delta(f)$ from $g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s)$ as

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s) e^{-j2\pi nT_s f}$$

- Hence, we have

$$G_\delta(f) = f_s \sum_{n=-\infty}^{\infty} G(f - nf_s) = \sum_{n=-\infty}^{\infty} g(nT_s) e^{-j2\pi nT_s f}$$

- From the Fourier transform pair of the sampled signal, we can write

$$G_\delta(f) = f_s G(f) + f_s \sum_{m=-\infty, m\neq 0}^{\infty} G(f - mf_s)$$

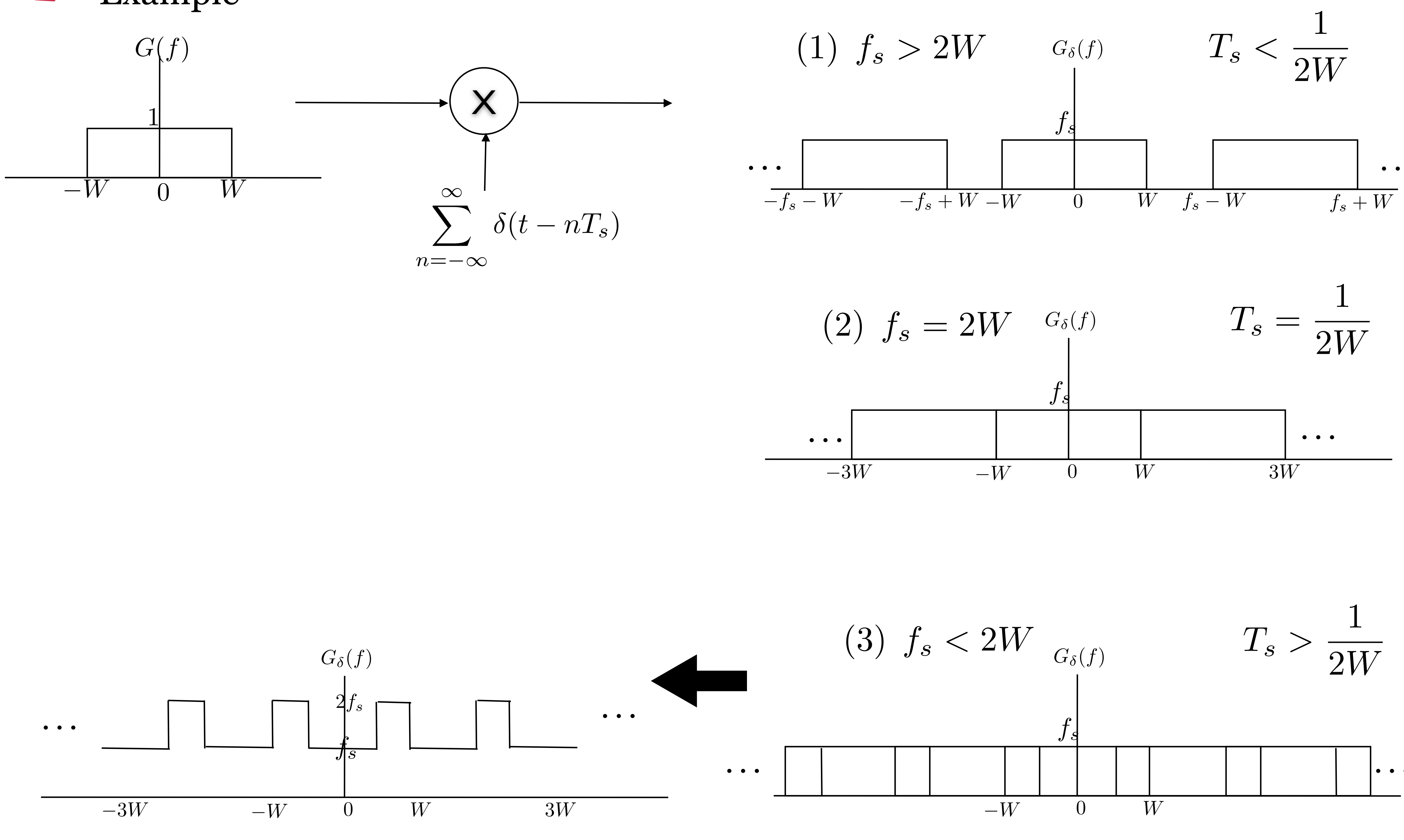
- From this expression, we find that

$$G(f) = \frac{1}{f_s} G_\delta(f), \qquad -W \leq f \leq W, \text{ if } f_s \geq 2W$$

  - or

$$G(f) = \frac{1}{f_s} \sum_{n=-\infty}^{\infty} g(nT_s) \exp(-j2\pi n T_s f), \qquad \text{for } -W \leq f \leq W$$

Example



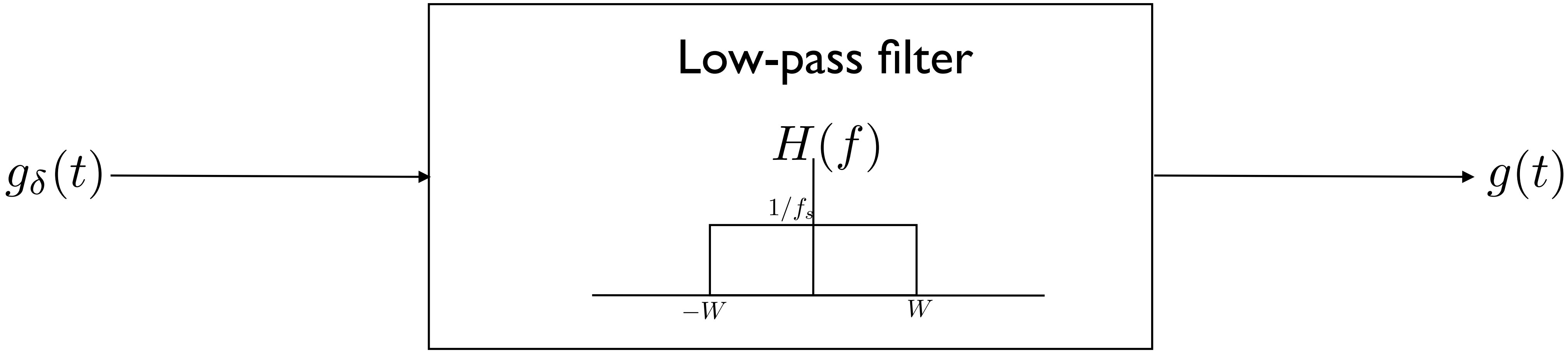
G(f)
1
−W
0
W
X
∞
Σ
n=−∞
δ(t − nT_s)
(1) f_s > 2W
G_δ(f)
T_s < 1/2W
f_s
−f_s − W
−f_s + W
−W
0
W
f_s − W
f_s + W
(2) f_s = 2W
G_δ(f)
T_s = 1/2W
f_s
−3W
−W
0
W
3W
(3) f_s < 2W
G_δ(f)
T_s > 1/2W
f_s
−W
0
W
G_δ(f)
2f_s
f_s
−3W
−W
0
W
3W

- Now let us answer two questions regarding the sampling process:

- Two questions
  - What are the restriction on $g(t)$ and T_s to allow perfect recovery of $g(t)$ from $g_\delta(t)$?
  - How is $g(t)$ recovered from $g_\delta(t)$?

- Answers
  - Sampling period T_s must be less than or equal to the inverse of two times signal bandwidth W, that is, $T_s \leq 1/2W$.
  - When the sampled signal is passed through the lowpass filter with the bandwidth W, the original signal can be perfectly recovered.

- From the previous example, we can reconstruct the original signal from the sampled signal such as



- Also note that the impulse function of the lowpass filter is given as

$$
\begin{aligned}
h(t) &= T_s \int_{-W}^{W} \exp(j2\pi f t)\, df \\
&= \frac{T_s}{j2\pi t}\left[e^{j2\pi W t} - e^{-j2\pi W t}\right] \\
&= 2WT_s \cdot \frac{e^{j2\pi W t} - e^{-j2\pi W t}}{j2 \cdot (2\pi W t)} \\
&= 2WT_s \mathrm{sinc}(2Wt) \\
&= \frac{2W}{f_s}\mathrm{sinc}(2Wt)
\end{aligned}
$$

If $f_s = 2W$,

$$h(t) = \mathrm{sinc}(2Wt)$$

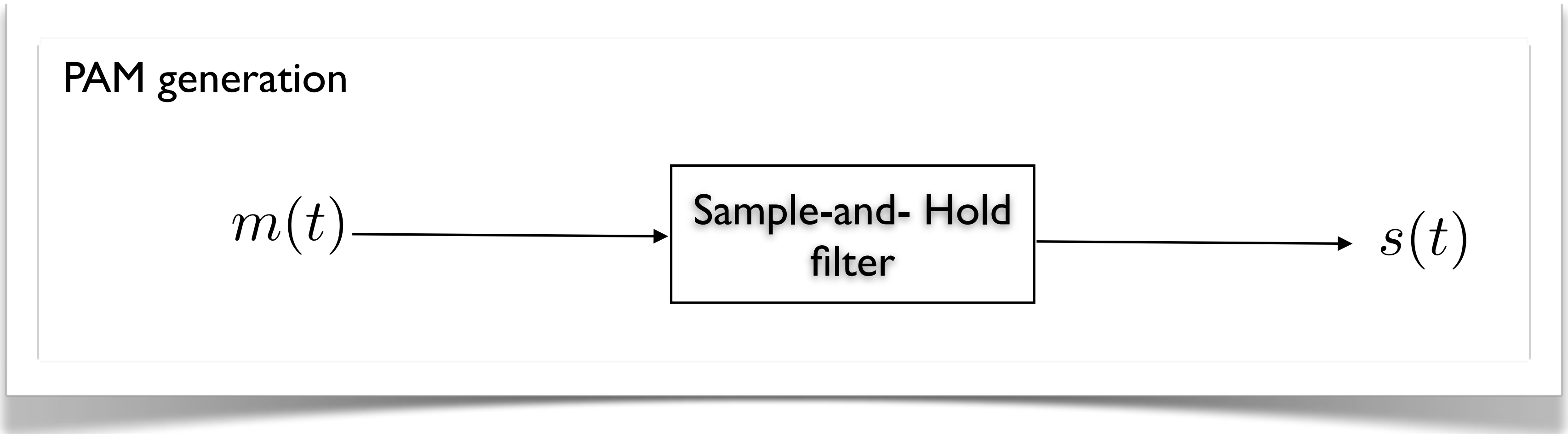
- Let us assume $f_s = 2W$, which is the minimum sampling rate that we can guarantee without distortion to recover the original signal. Then, we have

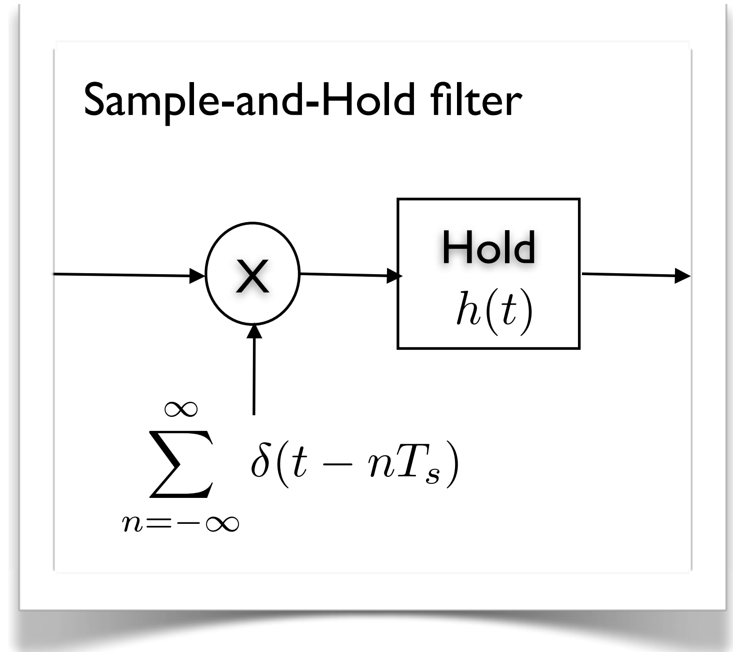
$$G(f) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} g(nT_s) \exp\left(-\frac{j\pi n f}{W}\right), \qquad \text{for } -W \le f \le W$$

- Its inverse Fourier transform is given as

$$\begin{aligned}
g(t) &= \int_{-\infty}^{\infty} G(f) e^{j2\pi f t}\, df \\
&= \int_{-W}^{W} \frac{1}{2W} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \exp\left(-\frac{j\pi n f}{W}\right) \exp(j2\pi f t)\, df \\
&= \frac{1}{2W} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \int_{-W}^{W} \exp\left[j2\pi f\left(t - \frac{n}{2W}\right)\right] df \\
&= \frac{1}{2W} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \frac{\exp\left[j2\pi W\left(t - \frac{n}{2W}\right)\right] - \exp\left[-j2\pi W\left(t - \frac{n}{2W}\right)\right]}{j2\pi\left(t - \frac{n}{2W}\right)} \\
&= \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \operatorname{sinc}\left[2W\left(t - \frac{n}{2W}\right)\right] \quad = \quad g_\delta(t) * \operatorname{sinc}(2Wt)
\end{aligned}$$

Pulse Amplitude Modulation





$$h(t) = \text{rect}\left(\frac{t-\frac{T}{2}}{T}\right) = \begin{cases} 1, & 0 < t < T \\ \frac{1}{2}, & t = 0,\ \ t = T, \\ 0, & \text{otherwise} \end{cases}$$

$$m_\delta(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\delta(t-nT_s)$$

$$s(t) = m_\delta(t) * h(t) = \sum_{n=-\infty}^{\infty} m(nT_s)h(t-nT_s)$$

Signal representation of PAM signal in time-domain

$$\begin{aligned}
m_\delta(t) * h(t) &= \int_{-\infty}^{\infty} m_\delta(t) h(t-\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} m(nT_s)\delta(\tau - nT_s) h(t-\tau)\, d\tau \\
&= \sum_{n=-\infty}^{\infty} m(nT_s) \int_{-\infty}^{\infty} \delta(\tau - nT_s) h(t-\tau)\, d\tau \\
&= \sum_{n=-\infty}^{\infty} m(nT_s) h(t - nT_s)
\end{aligned}$$

Signal representation of PAM signal in frequency-domain

$$\begin{aligned}
S(f) &= M_\delta(f) H(f) \\
&= \left(f_s \sum_{k=-\infty}^{\infty} M(f - kf_s) \right) H(f) \\
&= f_s \sum_{k=-\infty}^{\infty} M(f - kf_s) H(f)
\end{aligned}$$

where $H(f) = T \mathrm{sinc}(fT) \exp(-j\pi f T)$