Linear Stability Analysis

For 1-D

Governing Equation

- The governing equation (the non-local Cahn-Hilliard equation) is as follows :

$$\frac{\partial \phi(\mathbf{x},t)}{\partial t} = \Delta \left(f(\phi(\mathbf{x},t)) - \epsilon^2 \Delta \phi(\mathbf{x},t) \right) - \alpha \left(\phi(\mathbf{x},t) - \bar{\phi} \right)$$

for x in Ω, t in (0, T] where Ω = R^d (d = 1, 2, 3) is a domain.

- The Helmholtz free energy $F(\phi) = 0.25(\phi^2 - 1)^2$ with f=F′ and global minima $\phi = \pm 1$.

Coefficients in Governing equation

- $\bar{\phi} = \int_{\Omega} \phi d\mathbf{x}/|\Omega|$ is the average concentration of the initial condition

- α : inversely proportional to the square of the total chain length of the copolymer

- ε : the gradient energy coefficient.

Numerical Scheme

- We consider a spectral method with the discrete Fourier transform method. The linear and nonlinear terms treated implicitly and explicit, respectively.

$x_m = (m-1)L/M$ where m=1,…,M where M is even

$\phi^k = (\phi_1^k, \cdots, \phi_M^k)$, $\phi_m^k \approx \phi(x_m, k\Delta t)$

$$\hat{\phi}_p^k = \sum_{m=1}^{M} \phi_m^k e^{-ix_m\xi_p}, \quad \phi_m^k = \frac{1}{M} \sum_{p=1-M/2}^{M/2} \hat{\phi}_p^k e^{ix_m\xi_p}$$

where $\xi_p = 2\pi(p-1)/L$

Then, $$\hat{\phi}_p^{k+1} = \frac{\hat{\phi}_p^k - \Delta t \xi_p^2 \hat{g}_p^k + \Delta t \alpha \bar{\phi}}{1 + \alpha \Delta t + 2\Delta t \xi_p^2 + \epsilon^2 \Delta t \xi_p^4}$$

Linear Stability Analysis

- By the linear stability analysis, the solution of the governing equation is assumed to have the form of

$$\phi(x,t) = \bar{\phi} + \sum_{k=1}^{\infty} \beta_k(t) \cos(k\pi x)$$

where $|\beta_k(t)| \ll 1$.

- Substituting the above equation into the governing equation, we have

$$\frac{d\beta_k}{dt} = -(k\pi)^2[3\bar{\phi}^2 - 1 + \epsilon^2(k\pi)^2]\beta_k + \alpha\beta_k$$

Linear Stability Analysis

- The equation as in the last slide

$$\frac{d\beta_k}{dt} = -(k\pi)^2[3\bar{\phi}^2 - 1 + \epsilon^2(k\pi)^2]\beta_k + \alpha\beta_k$$

is up to first order.

- The solution of above one is $\beta_k(t) = \beta_k(0)e^{\eta_k t}$ where

$$\eta_k = -(k\pi)^2[3\bar{\phi}^2 - 1 + \epsilon^2(k\pi)^2] - \alpha$$

is the growth rate.

Linear Stability Analysis

- Therefore, we have the maximal growth rate

$$k = \sqrt{1 - 3\bar{\phi}^2} / \left(\sqrt{2}\phi\epsilon\right)$$

- Next, the theoretical growth rate η_k is compared to the numerical growth rate by our proposed scheme with different values of k and α.

Numerical Test

- The numerical growth rate is defined by

$$\eta_k = \frac{1}{T} \log \left(\frac{\max_i |\phi_i^n|}{\max_i |\phi_i^0|} \right)$$

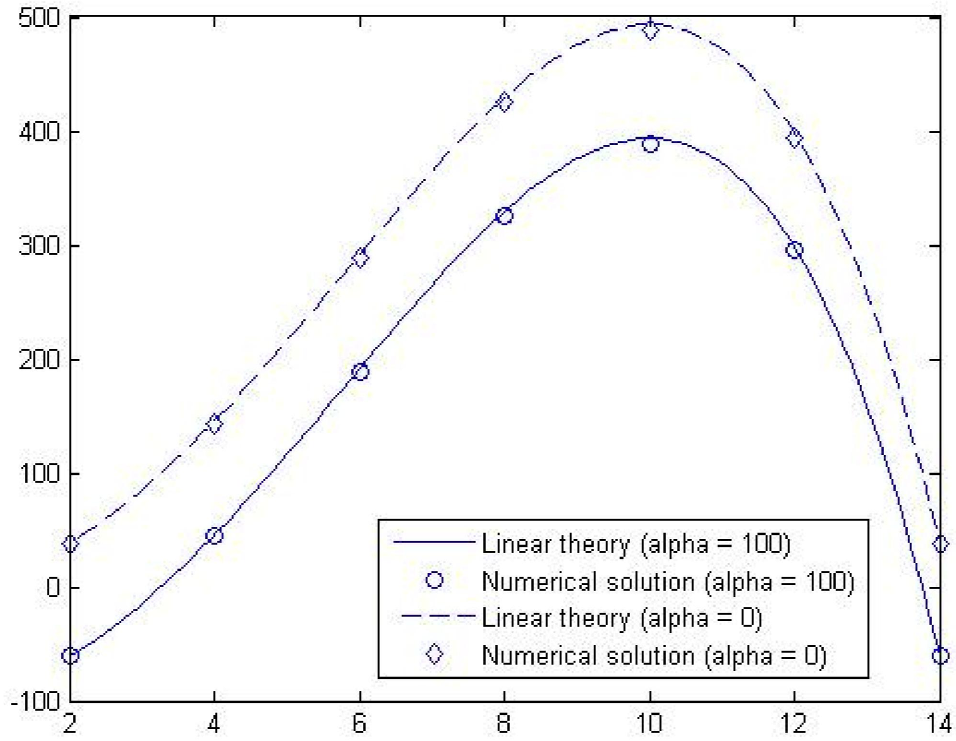
- For the numerical test, we let $\bar{\phi} = 0$ and an initial condition $\phi(x,0) = 0.01 \cos(k\pi x)$ with $\epsilon = 1/\left(10\sqrt{2}\pi\right)$, $\Delta t = 10^{-6}$ and $h = 0.01$ until $T = 10^{-4}$.

Numerical Results

- Figure in the next slide will suggests that theoretical values from the linear analysis (solid and dashed line) and numerical values (circle and diamond) are in good agreement.

- Moreover, the result shows that k = 10 is the maximal growth rate which is consistent with the result from the theory.

Figure



500
400
300
200
100
0
-100
2
4
6
8
10
12
14
Linear theory (alpha = 100)
Numerical solution (alpha = 100)
Linear theory (alpha = 0)
Numerical solution (alpha = 0)