Wireless Communications (ITC731)

Lecture Note 9

7-May-2013

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Summary

- Spatial multiplexing MIMO
  - Channel capacity with Random Channel Coefficient
  - Layered space-time MIMO architecture

Capacity of MIMO Fast (or Block) Fading Channels

$$C = E\left[W \sum_{i=1}^{r} \log_2\left(1 + \lambda_i \frac{P}{n_T \sigma^2}\right)\right]$$

where $r = \min(n_T, n_R)$

$$= E\left\{W \log_2 \det\left[\left(\mathbf{I}_r + \frac{P}{\sigma^2 n_T}\mathbf{Q}\right)\right]\right\}$$

where

$$\mathbf{Q} = \begin{cases} \mathbf{H}\mathbf{H}^H, & n_R \le n_T \\ \mathbf{H}^H\mathbf{H}, & n_T \le n_R \end{cases}$$

$$E\left[\sum_{i=1}^{r}\log_2\left(1+\lambda_i\frac{P}{n_T\sigma^2}\right)\right]=\sum_{i=1}^{r}E\left[\log_2\left(1+\lambda_i\frac{P}{n_T\sigma^2}\right)\right]$$

Wishart distribution with parameter r

$$p_{\lambda,\mathrm{ordered}}(\lambda_1,...,\lambda_m)=K_{m,n}^{-1}e^{-\sum_i\lambda_i}\prod_i\lambda_i^{n-m}\prod_{i<j}(\lambda_i-\lambda_j)^2,$$

$$\lambda_1\geq\cdots\geq\lambda_m$$

where $m=\max(n_T,n_R)$

$$n=\min(n_T,n_R)$$

$$K_{m,n}:\text{normalizing factor}$$

Density of unordered eigenvalues

$$p_\lambda(\lambda_1, ..., \lambda_m) = (m! K_{m,n})^{-1} e^{-\sum_i \lambda_i} \prod_i \lambda_i^{n-m} \prod_{i<j} (\lambda_i - \lambda_j)^2.$$

$$E\left[\sum_{i=1}^{r} \log_2\left(1 + \lambda_i \frac{P}{n_T \sigma^2}\right)\right] = \sum_{i=1}^{r} E\left[\log_2\left(1 + \lambda_i \frac{P}{n_T \sigma^2}\right)\right]$$

$$= \sum_{i=1}^{m} E\left[\log_2\left(1 + \lambda_i \frac{P}{n_T \sigma^2}\right)\right]$$

$$= mE\left[\log_2\left(1 + \lambda_1 \frac{P}{n_T \sigma^2}\right)\right]$$

$$p_{\lambda_1}(\lambda_1) = \int \cdots \int p_\lambda(\lambda_1, ..., \lambda_m)\, d\lambda_2 \cdots d\lambda_m.$$

Note that $\prod_{i<j}(\lambda_i - \lambda_j)$ is the determinant of a Vandermonde matrix

$$D(\lambda_1, ..., \lambda_m) = \begin{bmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_m \\ \vdots & \vdots & \vdots \\ \lambda_1^{m-1} & \cdots \lambda_m^{m-1} & \end{bmatrix}$$

and we can write

$$p_\lambda(\lambda_1, ..., \lambda_m) = (m!K_{m,n})^{-1} \det(D(\lambda_1, ..., \lambda_m))^2 \prod_i \lambda_i^{n-m} e^{-\lambda_i}$$

Consider two real valued function with inner product

$$< f, g >= \int_0^\infty f(\lambda)g(\lambda)\lambda^{n-m}e^{-\lambda}\, d\lambda.$$

Apply the Gram-Schmidt orthogonalization procedure to the sequence:

$$1, \lambda, \lambda^2, ..., \lambda^{m-1}$$

with orthogonal functions defined as

$$\int_0^\infty \phi_i(\lambda)\phi_j(\lambda)\lambda^{n-m}e^{-\lambda}\, d\lambda = \delta_{ij}.$$

With row operations we can transform $D(\lambda_1, ..., \lambda_m)$ into

$$\tilde{D}(\lambda_1, ..., \lambda_m) = \begin{bmatrix} \phi_1(\lambda_1) & \cdots & \phi_1(\lambda_m) \\ \vdots & \vdots & \vdots \\ \phi_m(\lambda_1) & \cdots & \phi_m(\lambda_m) \end{bmatrix}$$

The determinant of D then equals the determinant of $\tilde{D}$!

$$\det\left[\tilde{D}(\lambda_1, ..., \lambda_m)\right] = \sum_a (-1)^{\mathrm{per}(a)} \prod_i \tilde{D}_{a,i}$$

$$= \sum_a (-1)^{\mathrm{per}(a)} \prod_i \phi_{a_i}(\lambda_i)$$

where the summation is over all permutation of $\{1, ..., m\}$, and $\mathrm{per}(a)$ is 0 or 1 depending on the permutation a being even or odd.

Thus

$$p_\lambda(\lambda_1, ..., \lambda_m) = C_{m,n} \sum_{a,b} (-1)^{\mathrm{per}(a)+\mathrm{per}(b)} \prod_i \psi_{a_i}(\lambda_i)\psi_{b_i}(\lambda_i)\lambda_i^{n-m} e^{-\lambda_i}$$

Integrating over $\lambda_1, ..., \lambda_m$, we get

$$p_{\lambda_1}(\lambda_1) = C_{m,n} \sum_{a,b} (-1)^{\mathrm{per}(a)+\mathrm{per}(b)} \psi_{a_1}(\lambda_1)\psi_{b_1}(\lambda_1)\lambda_1^{n-m} e^{-\lambda_1} \prod_{i \geq 2} \delta_{a_i b_i}$$

$$= C_{m,n}(m-1)! \sum_{i=1}^{m} \psi_i(\lambda_1)^2 \lambda_1^{n-m} e^{-\lambda_1}$$

if $a_i = b_i$ for $i \geq 2$ then $a_1 = b_1$ and thus $a = b$

$$\int_0^\infty p_{\lambda_1}(\lambda_1)\, d\lambda_1 = 1 \implies C_{m,n} = \frac{1}{m!}$$

Then we have

$$p_{\lambda_1}(\lambda_1) = \frac{1}{m}\sum_{i=1}^{m}\psi_i(\lambda_1)^2\lambda_1^{n-m}e^{-\lambda_1}$$

Observe now that the Gram-Schmidt orthonormalization yields

$$\psi_{k+1}(\lambda) = \left[\frac{k!}{(k+n+m)!}\right]^{1/2} L_k^{n-m}(\lambda), \quad k = 0, ..., m-1.$$

where

$$L_k^{n-m}(\lambda) = \frac{1}{k!}e^x x^{m-n}\frac{d^k}{dx^k}(e^{-x}x^{n-m+k})$$

is the Laguerre polynomial of order k.

Capacity of fast Rayleigh fading channel with power constraint P

$$\frac{\bar{C}}{W} = \int_0^\infty \log_2 (1 + P\lambda/n_T) \sum_{k=0}^{m-1} \frac{k!}{(k+n+m)!} \left[L_k^{n-m}(\lambda)\right]^2 \lambda^{n-m} e^{-\lambda} \, d\lambda$$

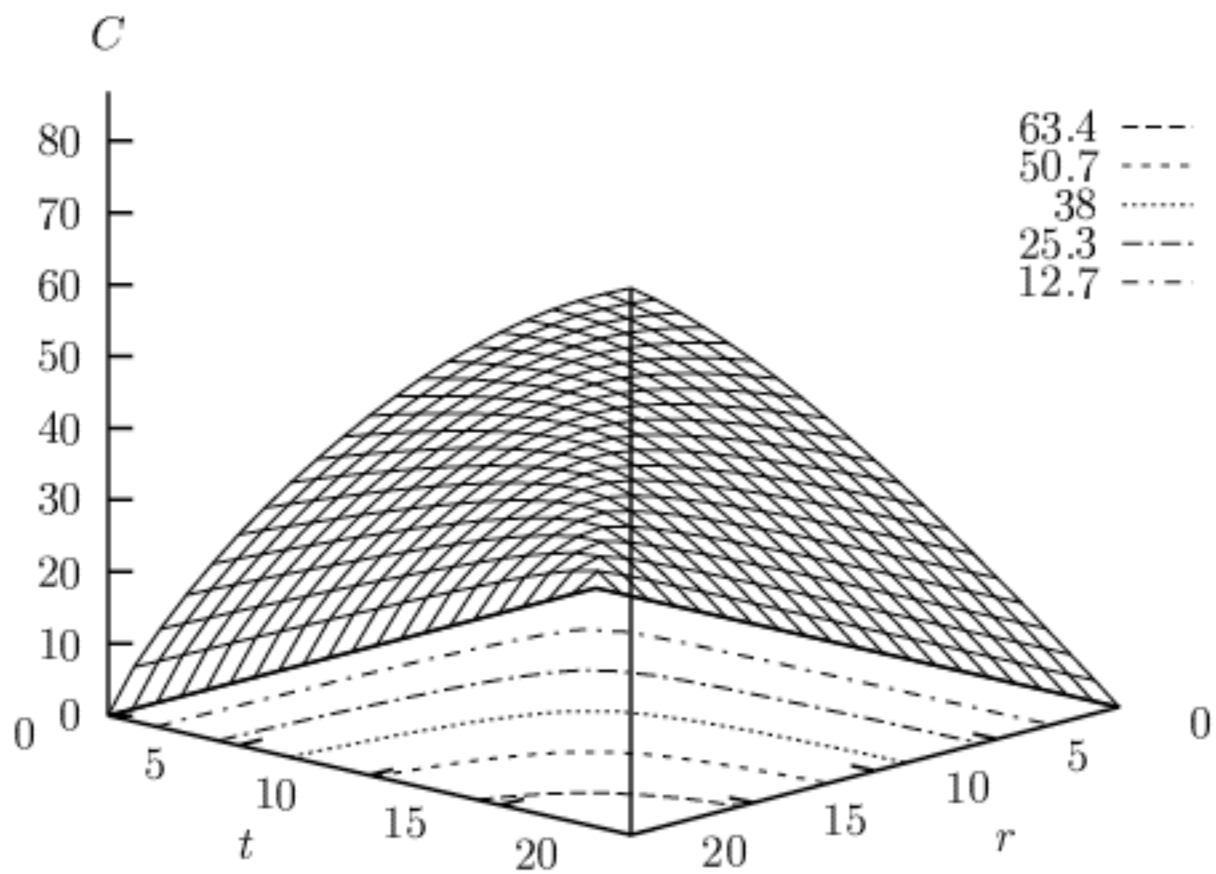
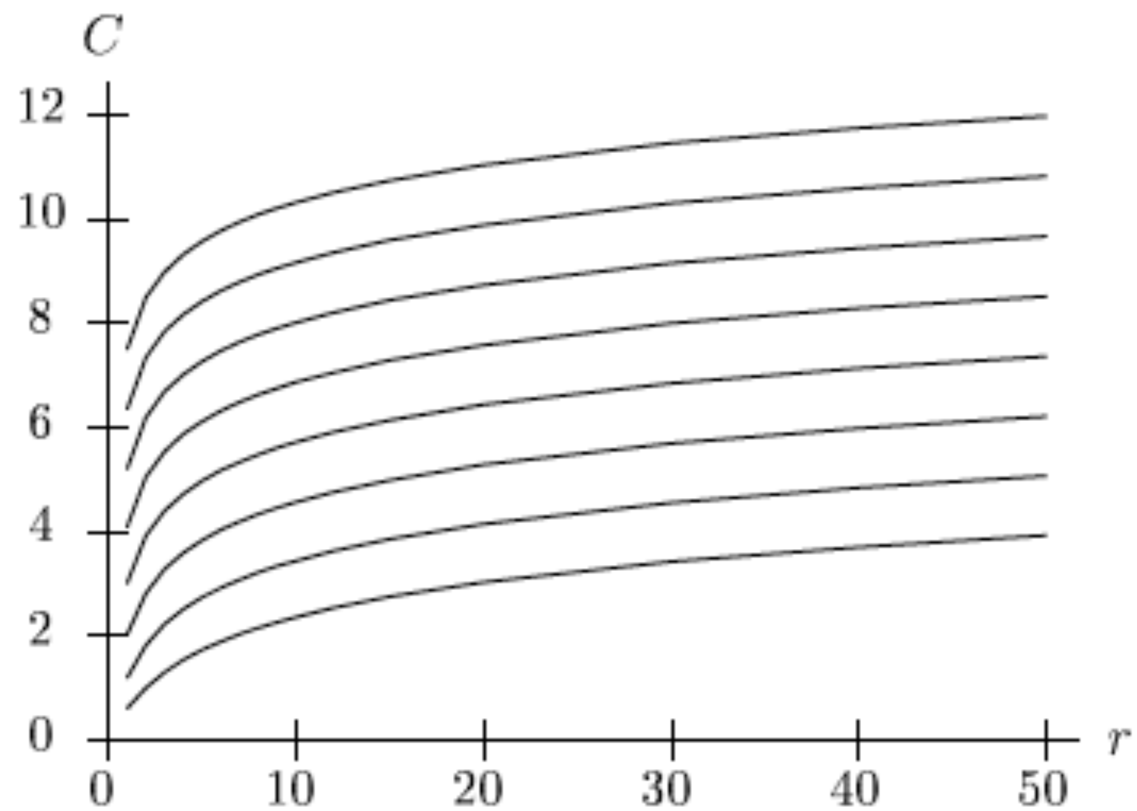


Figure 1: Capacity (in nats) vs. r and t for $P = 20$dB

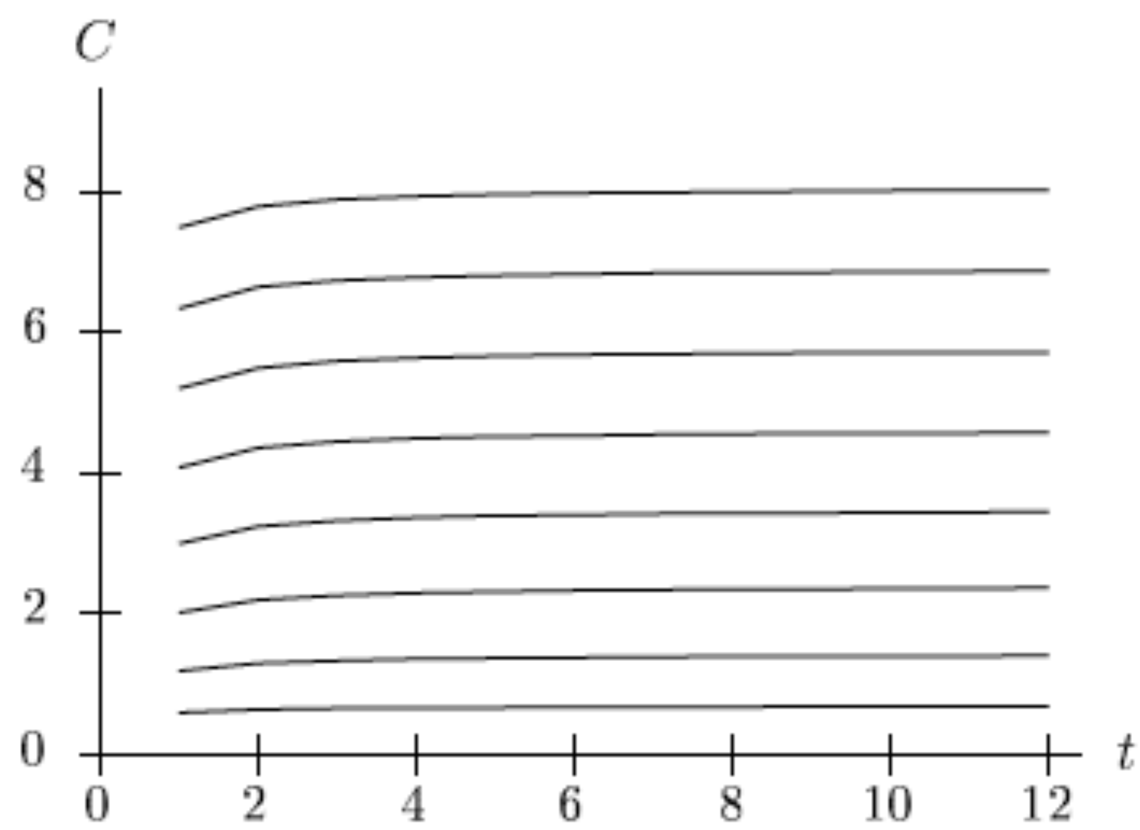
[E. Telatar, "Capacity of multi-antenna Gaussian channels", European Transactions on Telecommunications, vol. 10, No. 6, Nov./Dec. 1999, pp. 585-599]



The value of the capacity (in nats) as found from (9) vs. r for $0\text{dB} \leq P \leq 35\text{dB}$ in 5dB increments.

Figure 2: Capacity vs. r for $t = 1$ and various values of P.

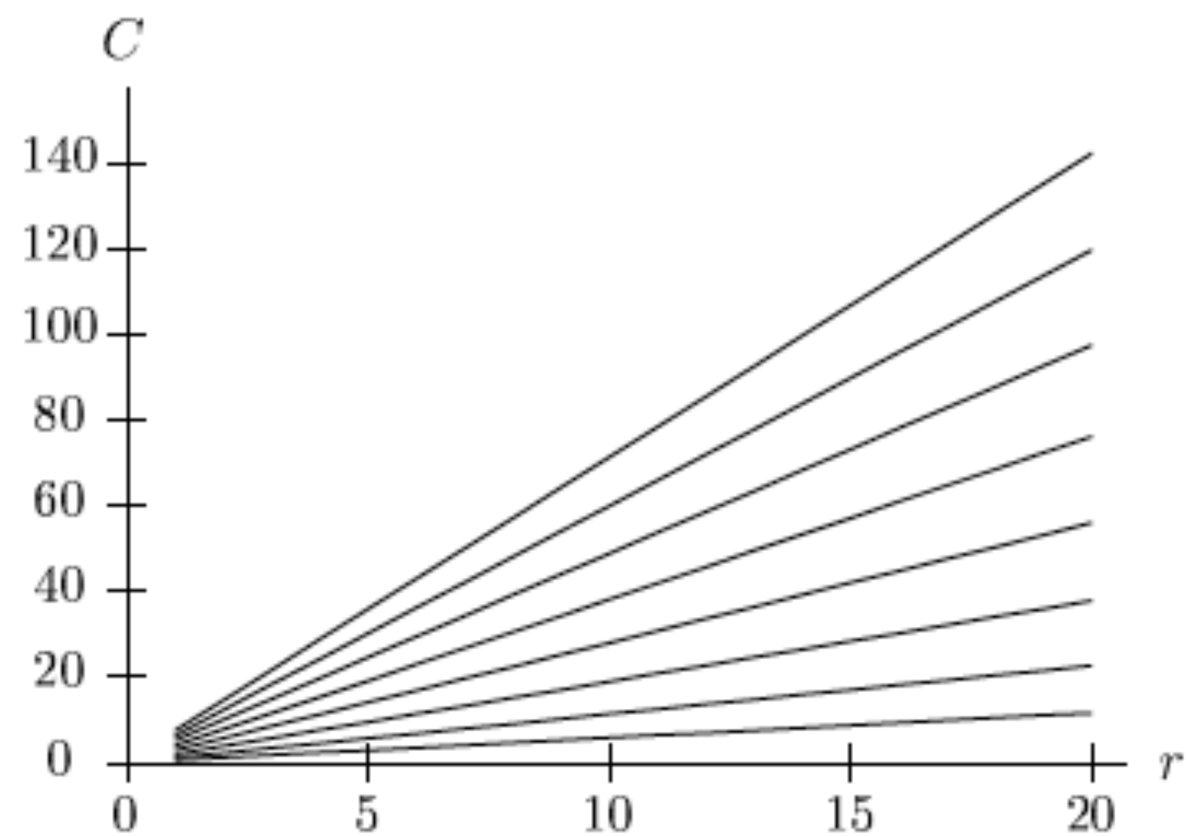
[E. Telatar, "Capacity of multi-antenna Gaussian channels", European Transactions on Telecommunications, vol. 10, No. 6, Nov./Dec. 1999, pp. 585-599]



The value of the capacity (in nats) as found from (10) vs. t for $0\text{dB} \leq P \leq 35\text{dB}$ in 5dB increments.

Figure 3: Capacity vs. t for $r = 1$ and various values of P.

[E. Telatar, "Capacity of multi-antenna Gaussian channels", European Transactions on Telecommunications, vol. 10, No. 6, Nov./Dec. 1999, pp. 585-599]



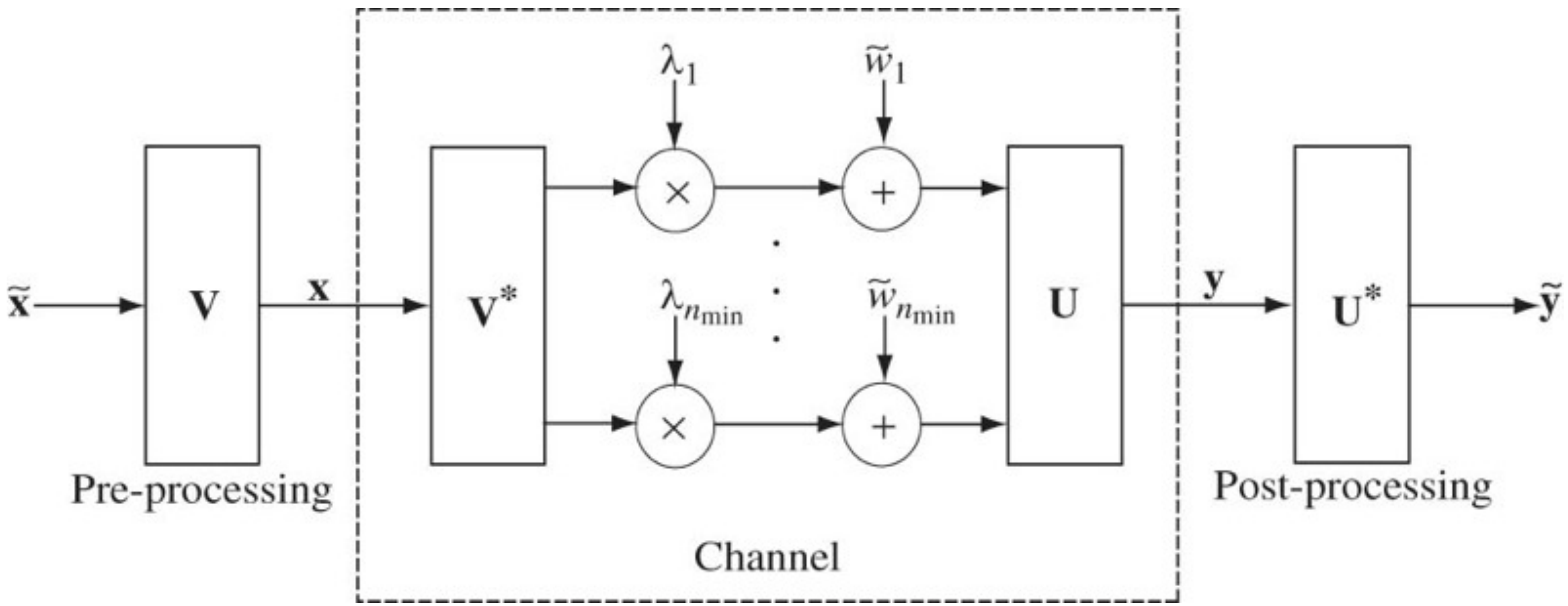
The value of the capacity (in nats) as found from (11) vs. r for $0\text{dB} \leq P \leq 35\text{dB}$ in 5dB increments.

Figure 4: Capacity vs. r for $r = t$ and various values of P.

[E. Telatar, "Capacity of multi-antenna Gaussian channels", European Transactions on Telecommunications, vol. 10, No. 6, Nov./Dec. 1999, pp. 585-599]

MIMO Capacity via SVD

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w}$$
$$\mathbf{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^H$$



Capacity is achieved by water-filling over the eigenmodes of the channel matrix H.

[Tse and Viswanathan, *Fundamentals of Wireless Communication*, Cambridge press]

Rank and Condition Number

At high SNR, equal power allocation is optimal.

$$C \approx \sum_{i=1}^{r} \log\left(1 + \frac{P\lambda_i^2}{rN_0}\right)$$

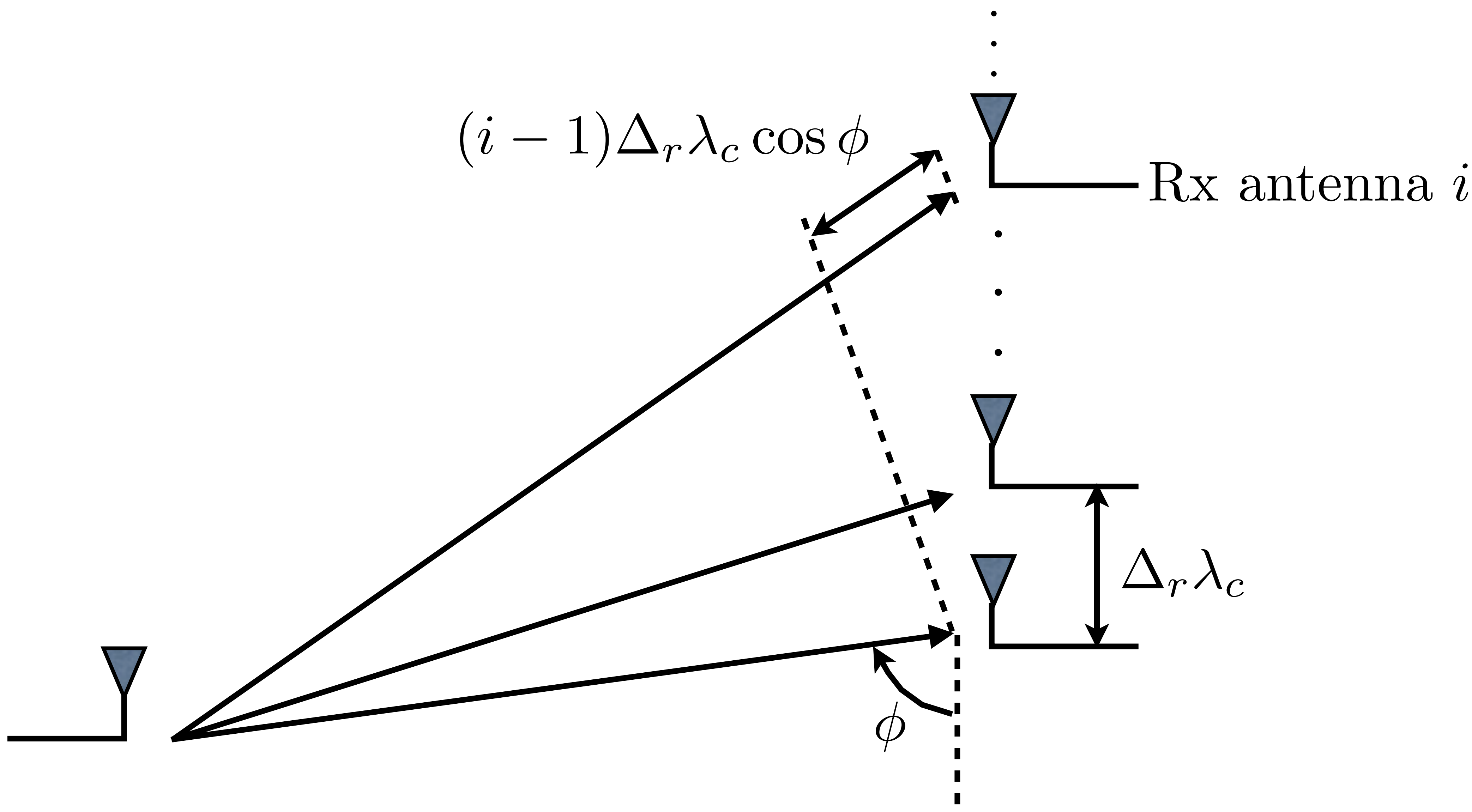
$$\approx r\log(\mathrm{SNR}) + \sum_{i=1}^{r} \log\left(\frac{\lambda_i^2}{k}\right)$$

where r is the number of nonzero λ_i, that is, the rank of $\mathbf{H}$.

The closer the condition number: $\dfrac{\max_i \lambda_i}{\min_i \lambda_i}$

to 1, the higher capacity.

SIMO with Line-of-Sight



$$\mathbf{y} = \mathbf{h}x + \mathbf{w}$$

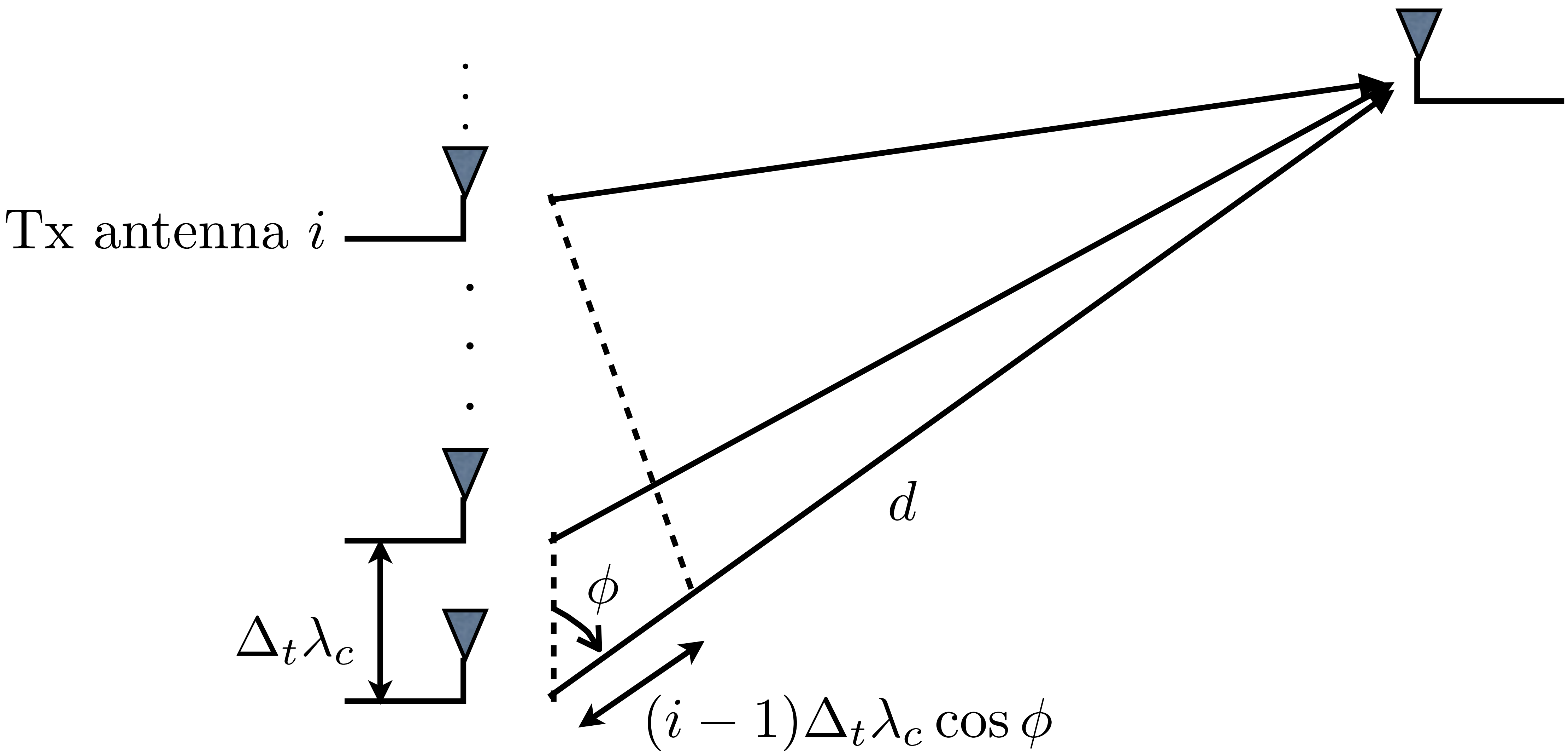
$$\Omega = \cos\phi$$

h is along the receive spatial signature in the direction:

$$e_r(\Omega) = \frac{1}{\sqrt{n_r}} \begin{bmatrix} 1 \\ \exp(-j2\pi\Delta_r\Omega) \\ \exp(-j2\pi 2\Delta_r\Omega) \\ \vdots \\ \exp(-j2\pi(n_r-1)\Delta_r\Omega) \end{bmatrix}$$

n_r-fold power gain

MISO over Line-Of-Sight



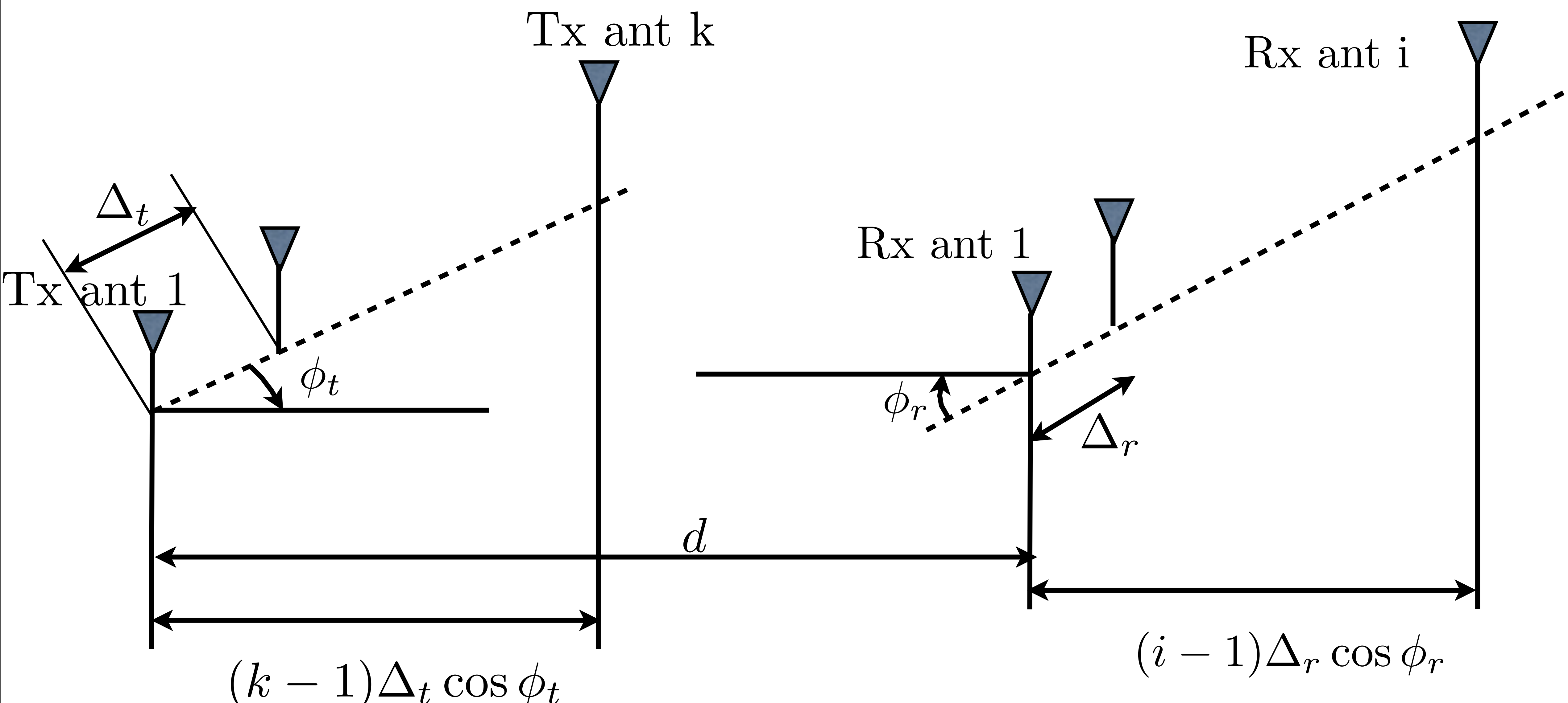
$\mathbf{h}$ is along the transmit spatial sequence in the direction $\Omega = \cos\phi$:

$\mathbf{h}$ is along the transmit spatial sequence in the direction $\Omega = \cos\phi$:

$$e_t(\Omega) = \frac{1}{\sqrt{n_t}} \begin{bmatrix} 1 \\ \exp(-j2\pi\Delta_t\Omega) \\ \exp(-j2\pi 2\Delta_t\Omega) \\ \vdots \\ \exp(-j2\pi(n_t-1)\Delta_t\Omega) \end{bmatrix}$$

n_t-fold power gain.

MIMO over Line-of-Sight



$$\mathbf{H} = G \cdot e_r(\Omega_r) e_t(\Omega_t)^*$$

$n_r n_t$-fold power gain.

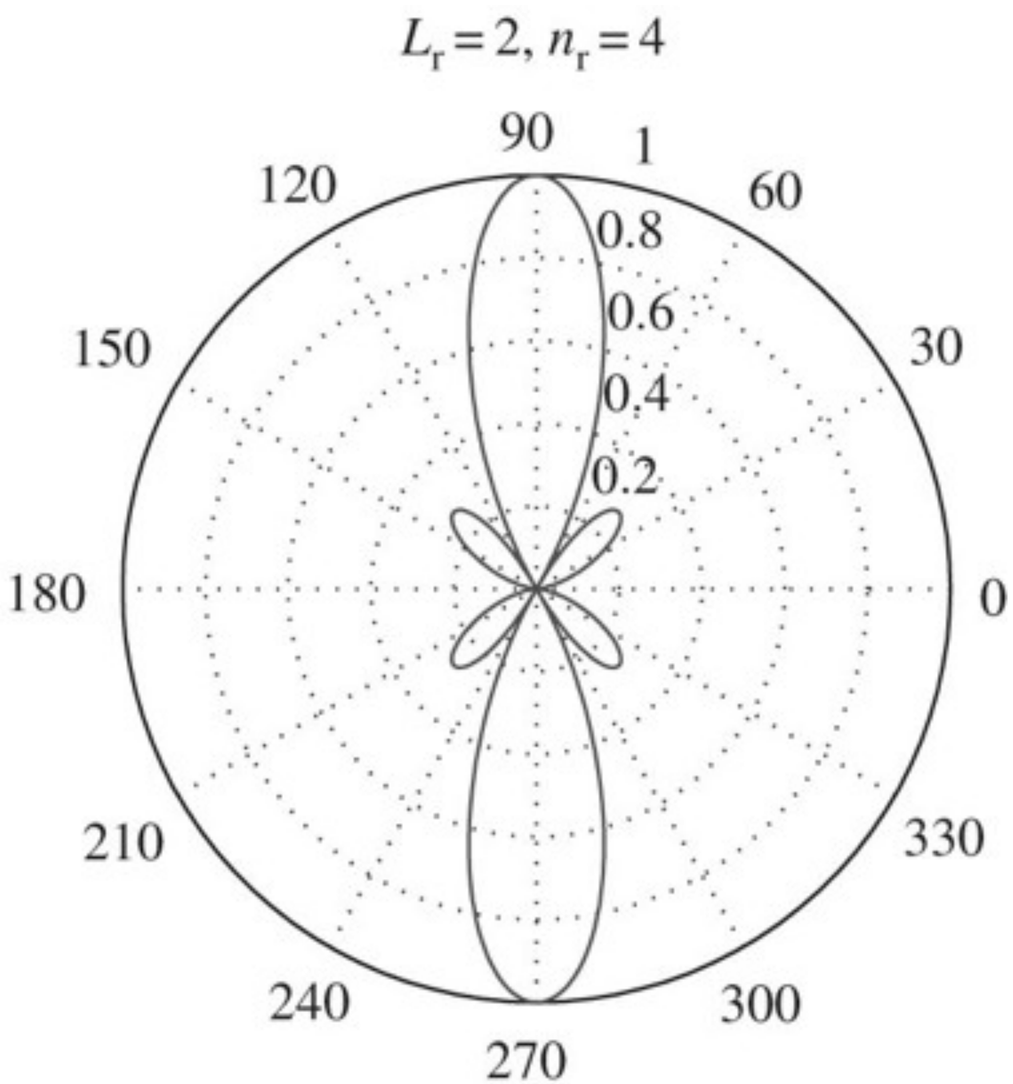
Rank 1, only one degree of freedom.

No spatial multiplexing gain.

Beamforming Patterns

- The receive beamforming pattern associated with $e_r(\Omega_0)$:

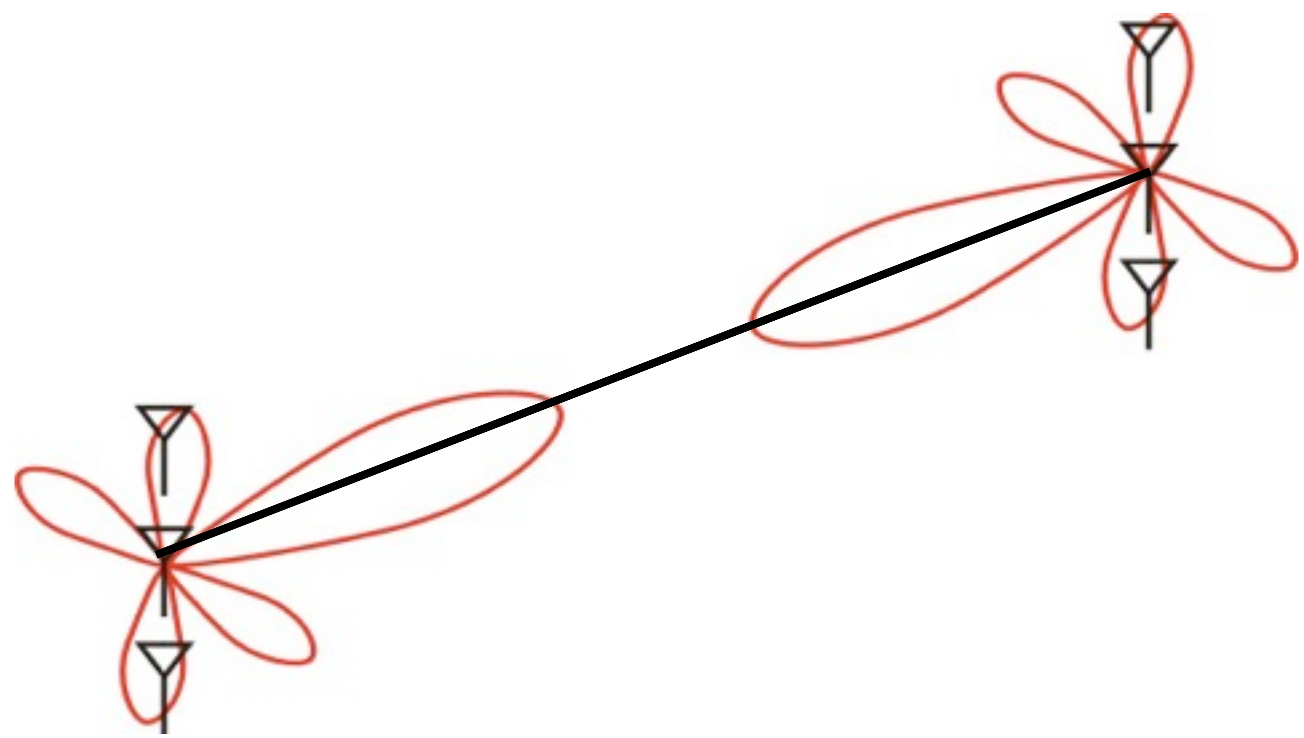
$$B_r(\Omega) := |e_r(\Omega_0)^* e_r(\Omega)|$$



Beamforming pattern gives the antenna gain in different directions!

[Tse and Viswanathan, *Fundamentals of Wireless Communication*, Cambridge press]

Line-of-Sight: Power Gain

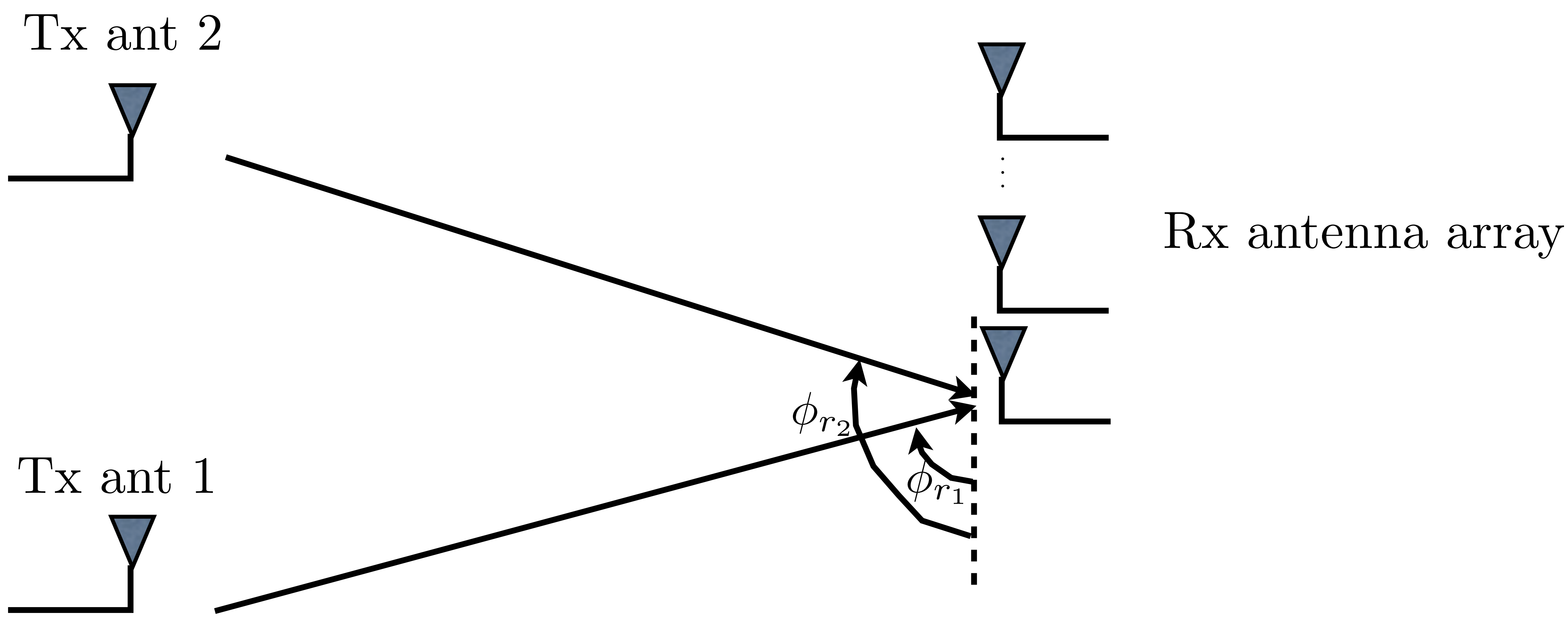


Energy is focused along a narrow beam.

Power gain but no degree-of-freedom gain.

[Tse and Viswanathan, *Fundamentals of Wireless Communication*, Cambridge press]

MIMO with TX Antenna Apart



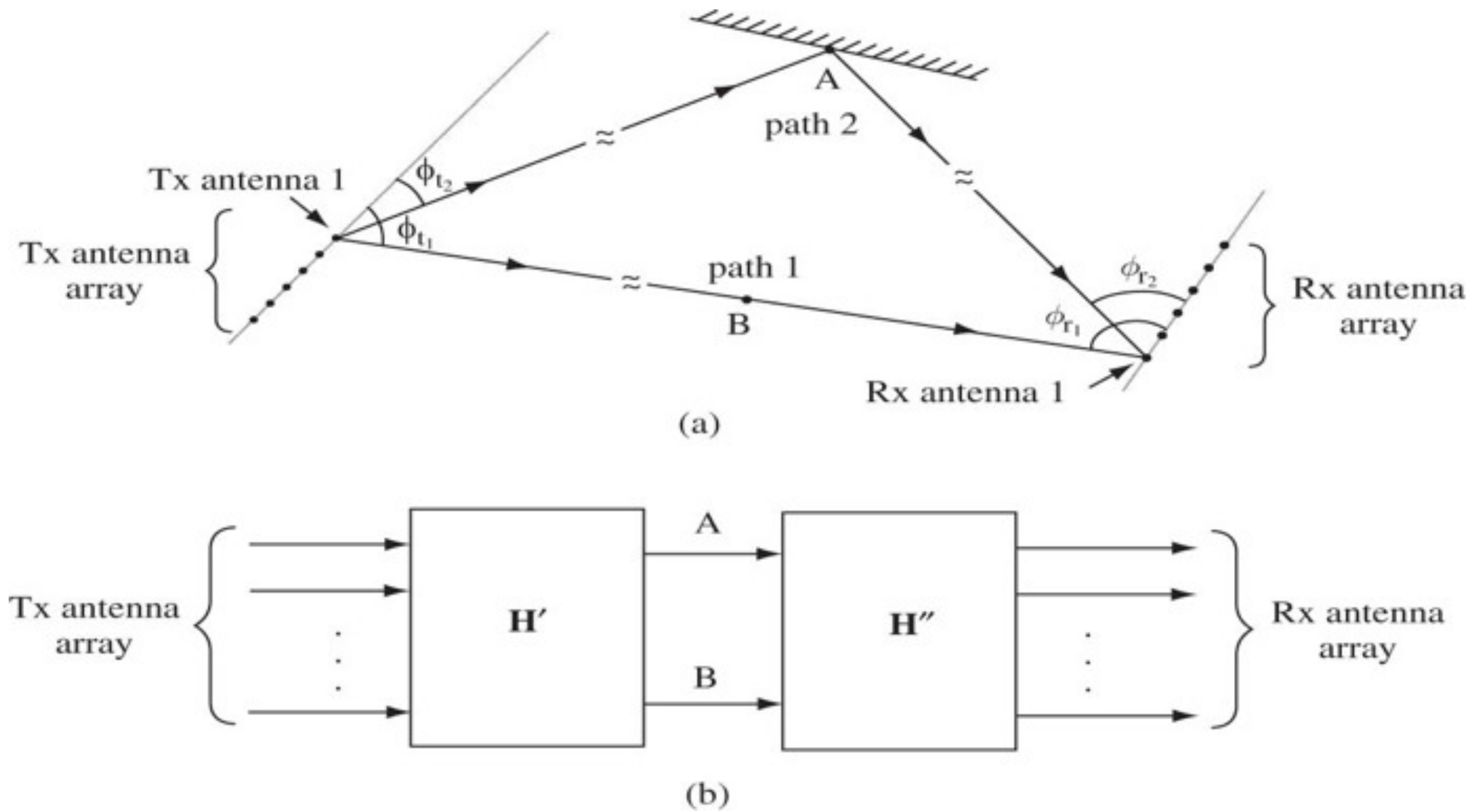
$\mathbf{h}_i$ is the receive spatial signature from Tx ant i along direction:

$$\Omega_i = G_i \cdot e_r(\Omega_i)$$

$$\mathbf{h}_i = G_i \cdot e_r(\Omega_i)$$

Two degrees of freedom if h_1 and h_2 are different

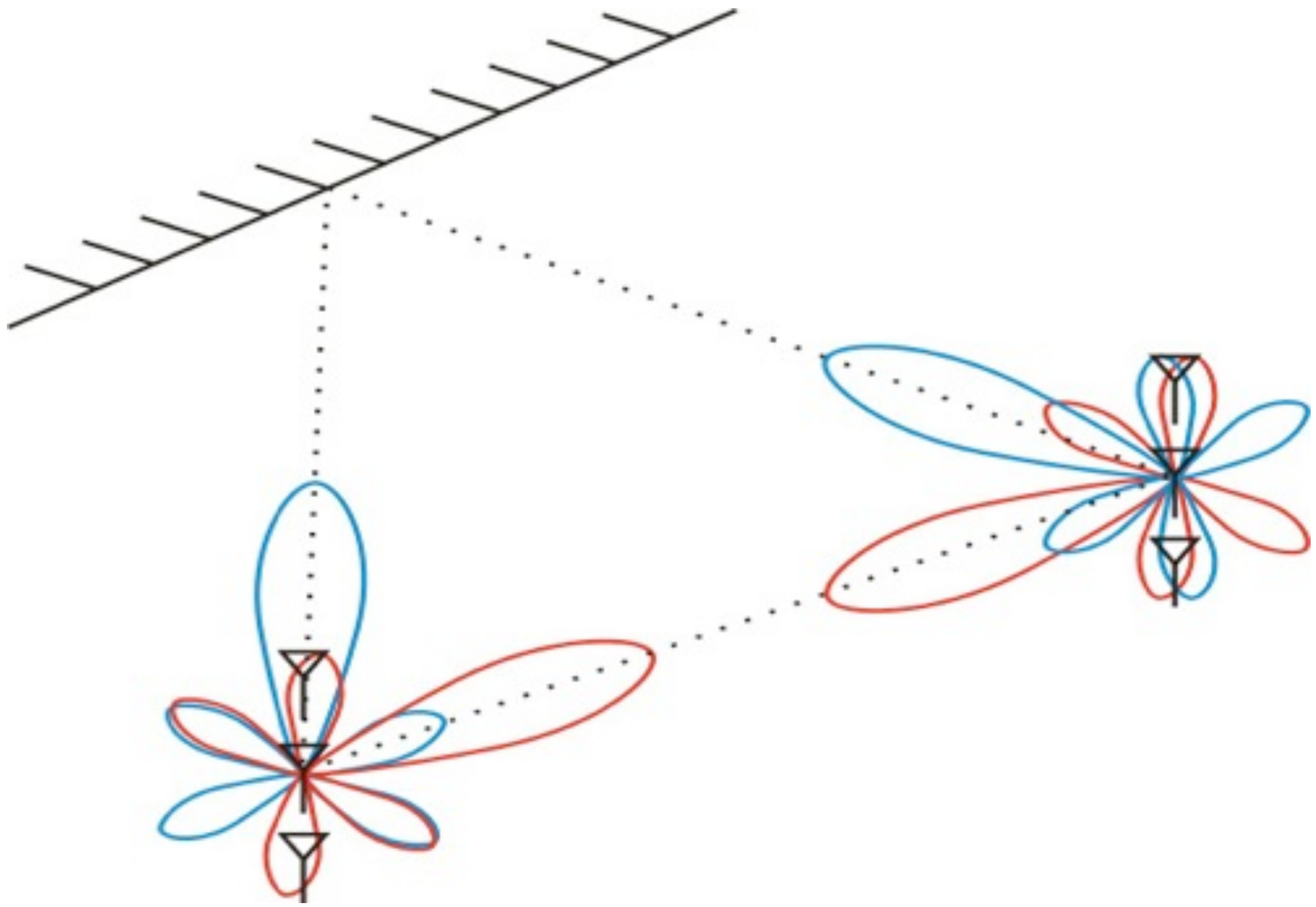
Example: Two-Path MIMO



A scattering environment provides multiple degrees of freedom even when the antennas are close together.

[Tse and Viswanathan, *Fundamentals of Wireless Communication*, Cambridge press]

[Beamforming patterns for two-path MIMO]



A scattering environment provides multiple degrees of freedom even when the antennas are close together.

[Tse and Viswanathan, *Fundamentals of Wireless Communication*, Cambridge press]

Beamforming Patterns

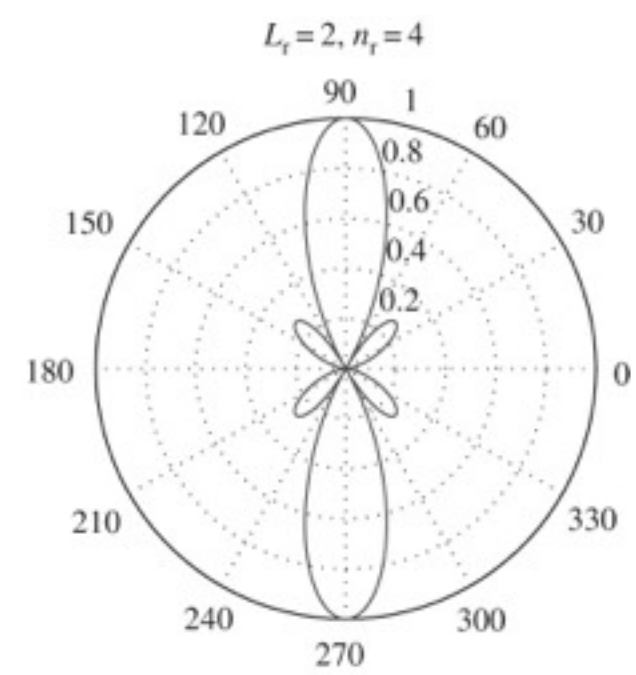
- The received beamforming pattern with $e_r(\Omega_0)$:

$$B_r(\Omega) := |e_r(\Omega_0)^* e_r(\Omega)|$$

L_r is the length of the antenna array, normalized to the carrier wavelength.

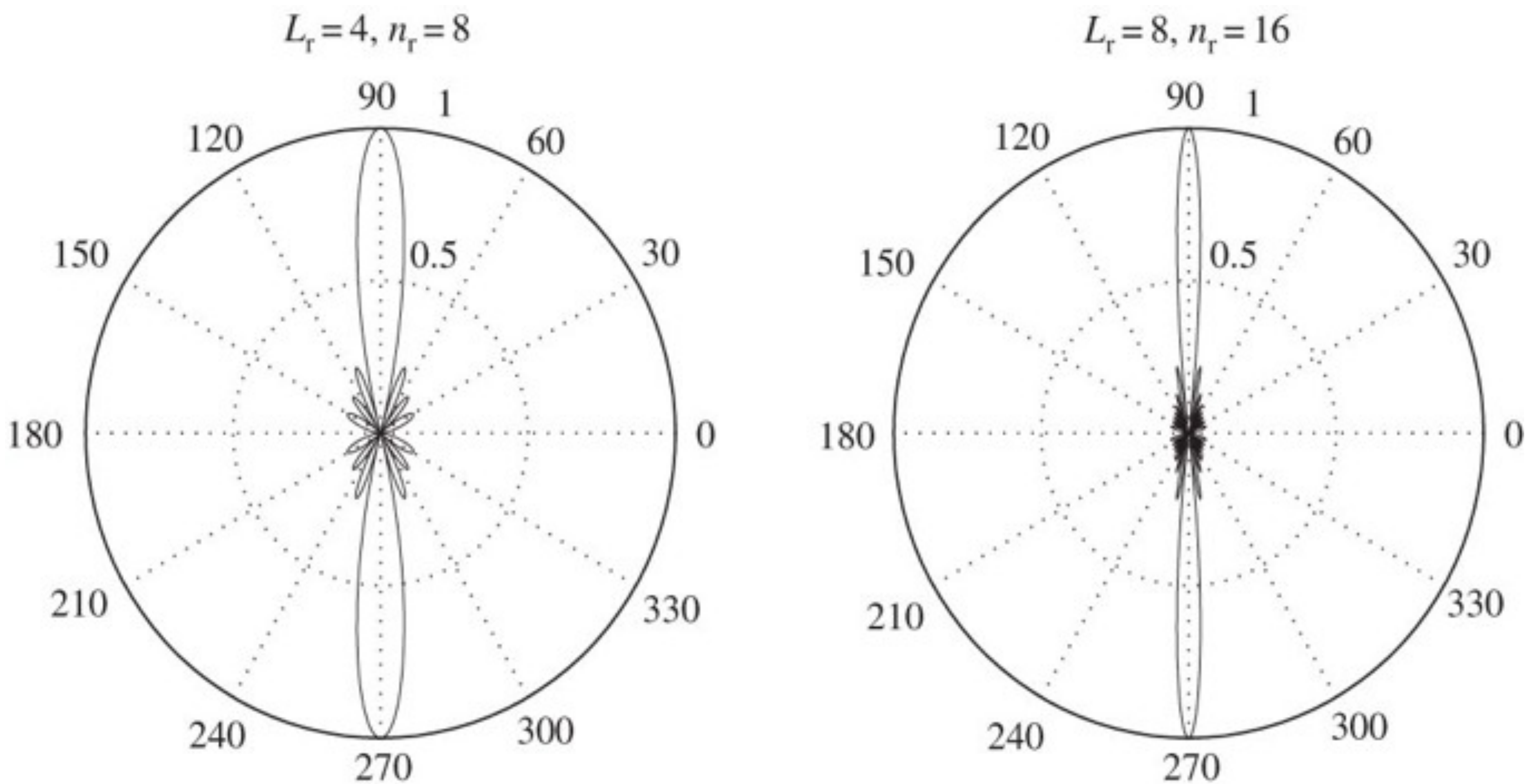
- Beamforming pattern gives the antenna gain in different directions.

- But it also tells us about angular resolvability.



[Tse and Viswanathan, *Fundamentals of Wireless Communication*, Cambridge press]

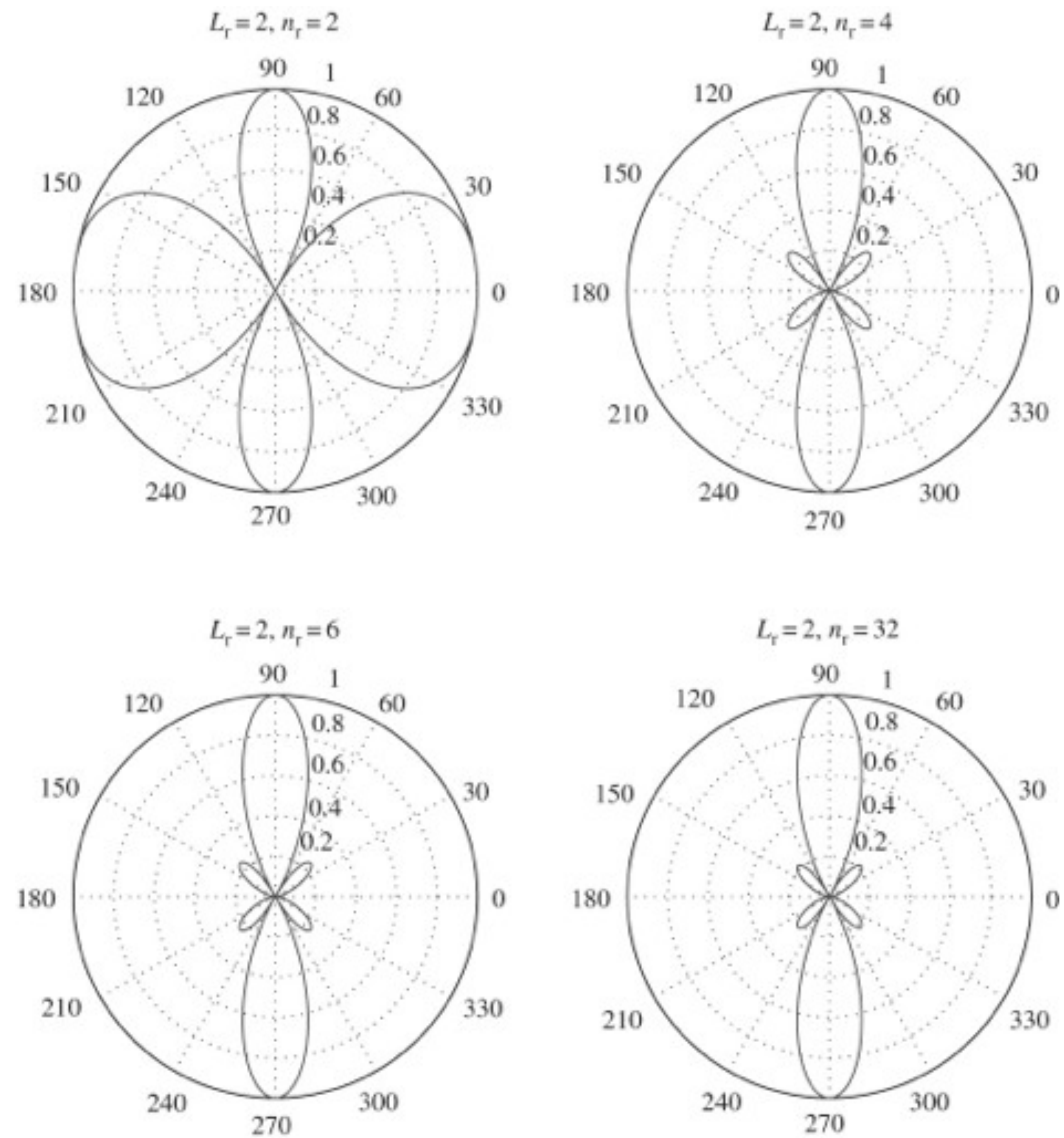
Angular Resolution

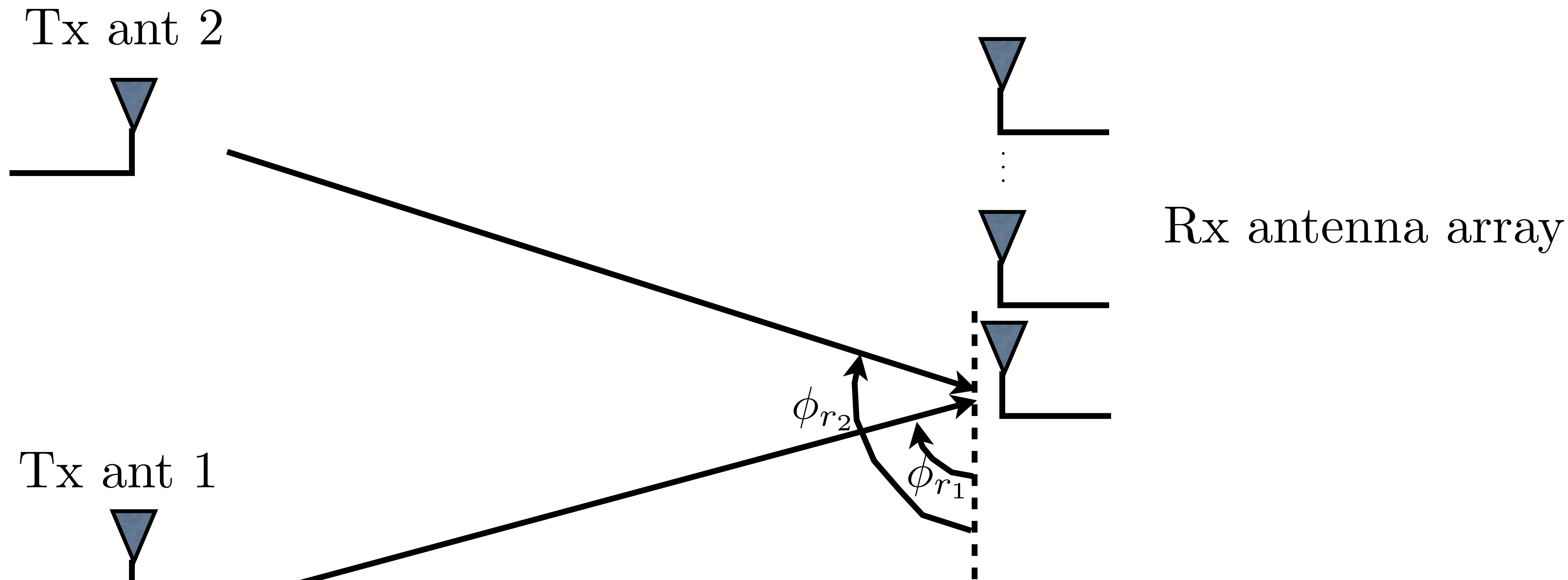


- Antenna array of length L_r provides angular resolution of $1/L_r$
  - Paths that arrive at angles closer is not very distinguishable.

Varying Antenna Separation

- Decreasing antenna separation beyond a half of wavelength ($\lambda/2$) has no impact on angular resolvability.



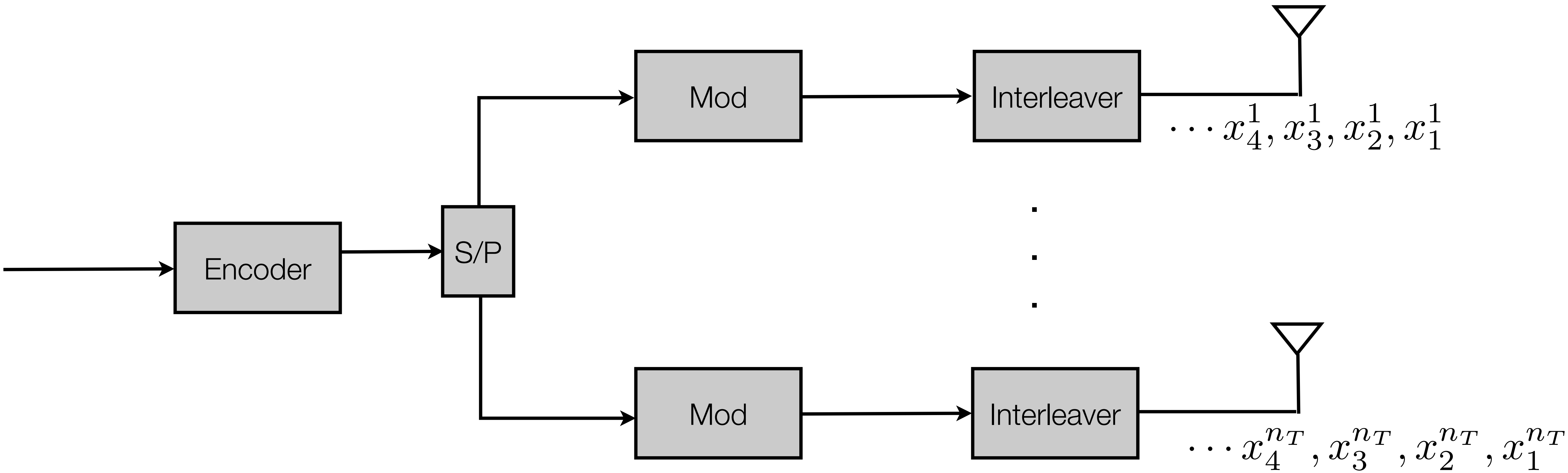


Channel **H** is well conditioned if

$$|\Omega_1 - \Omega_2| >> \frac{1}{L_r}$$

i.e., the signals from two TX antennas can be resolved.

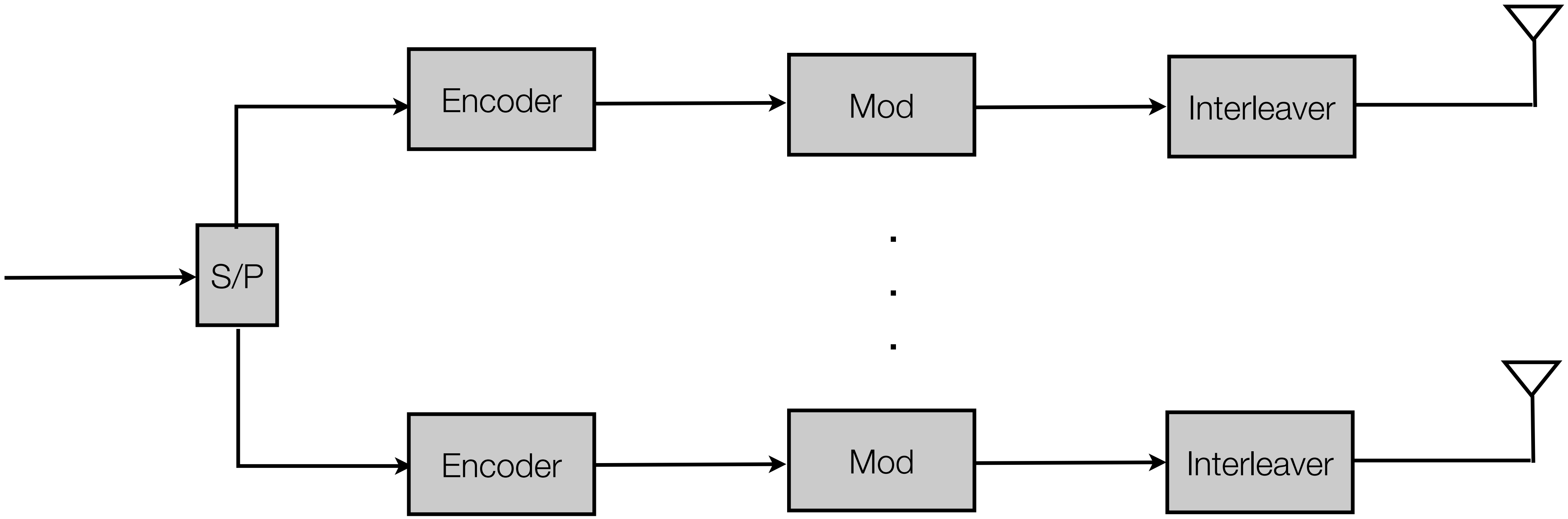
- Horizontal layered space-time (HLST) architecture



  - Transmission matrix

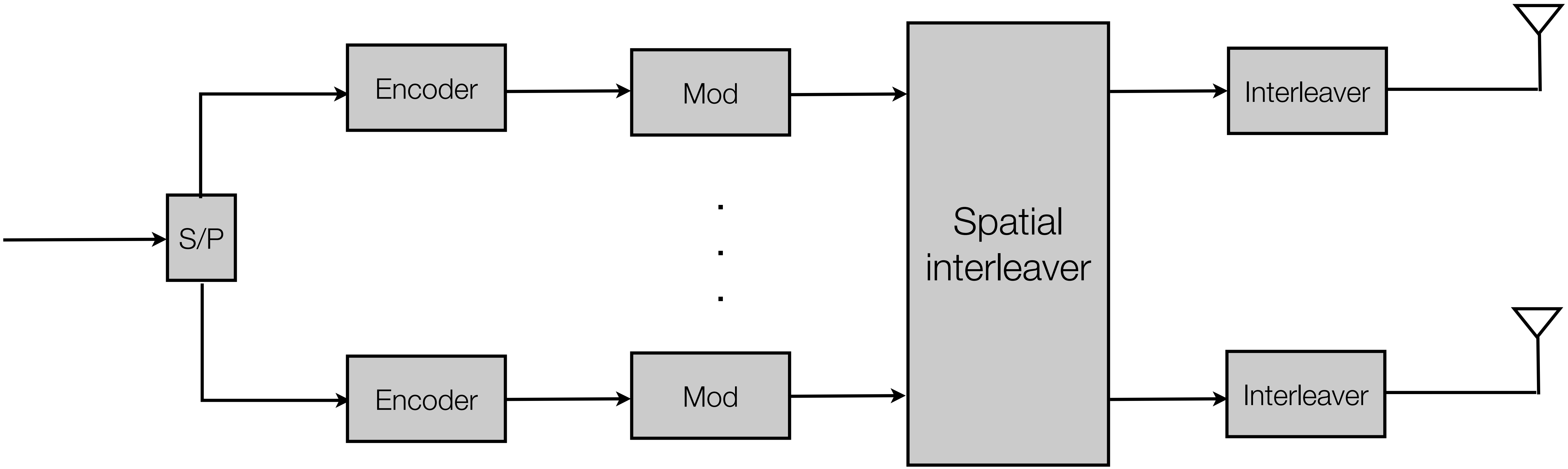
$$\mathbf{X} = \begin{bmatrix} x_1^1 & x_2^1 & x_3^1 & x_4^1 & \cdots \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & \cdots \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 & \cdots \end{bmatrix}$$

- HLST with encoder at each branch



  - Different coding in each sub-stream can be used.

Diagonal layered space-time (DLST) architecture



Spatial interleaving

$$\begin{bmatrix} x_1^1 & x_2^1 & x_3^1 & x_4^1 & x_5^1 & x_6^1 & \cdots \\ 0 & x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & \cdots \\ 0 & 0 & x_1^3 & x_2^3 & x_3^3 & x_4^3 & \cdots \end{bmatrix} \rightarrow \begin{bmatrix} x_1^1 & x_1^2 & x_1^3 & x_4^1 & x_4^2 & x_4^3 & \cdots \\ 0 & x_2^1 & x_2^2 & x_2^3 & x_5^1 & x_5^2 & \cdots \\ 0 & 0 & x_3^1 & x_3^2 & x_3^3 & x_6^1 & \cdots \end{bmatrix}$$

The diagonal layering introduces space diversity and thus achieves a better performance but with loss of spectral efficiency.

LST Receiver

- The signals transmitted from various antennas interfere with each other upon reception at the receiver.

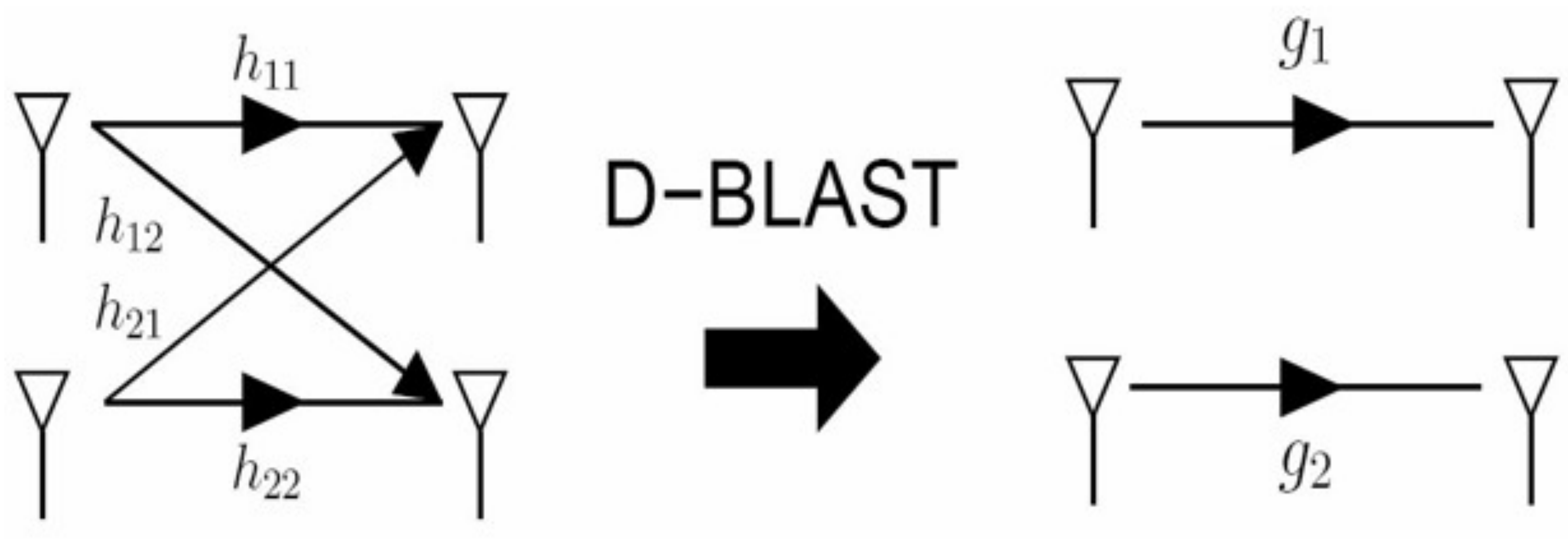
$$
\begin{aligned}
r_1 &= h_{11}x_1 + h_{12}x_2 + \cdots + h_{1n_T}x_{n_T} \\
r_2 &= h_{21}x_1 + h_{22}x_2 + \cdots + h_{2n_T}x_{n_T} \\
&\vdots
\end{aligned}
$$

$$
\mathbf{r}_t = \mathbf{H}\mathbf{x}_t + \mathbf{n_t}
$$

- At the receiver, we want to suppress and cancel the interference for the detection.

LST Architecture

- Parallel channel conversion



[Ref: Tse and Viswanathan, "Fundamentals of Wireless Communications", Cambridge Press]

QR decomposition

- Any $n_R \times n_T$ matrix $\mathbf{H}$, where $n_R \geq n_T$, can be decomposed as

$$\mathbf{H} = \mathbf{UR}$$

where

$$\mathbf{U} \quad : \quad n_R \times n_T \text{ unitary matrix}$$
$$\mathbf{R} \quad : \quad n_T \times n_T \text{ upper triangular matrix given as}$$

$$\mathbf{R} = \begin{bmatrix} R_{1,1} & R_{1,2} & \cdots & R_{1,n_T} \\ 0 & R_{2,2} & \cdots & R_{2,n_T} \\ 0 & 0 & \cdots & R_{2,n_T} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & R_{n_T,n_T} \end{bmatrix}$$

$$\mathbf{U}^T\mathbf{U} = \mathbf{I}$$

- Let us introduce n_T-component column matrix $\mathbf{y}$ defined as

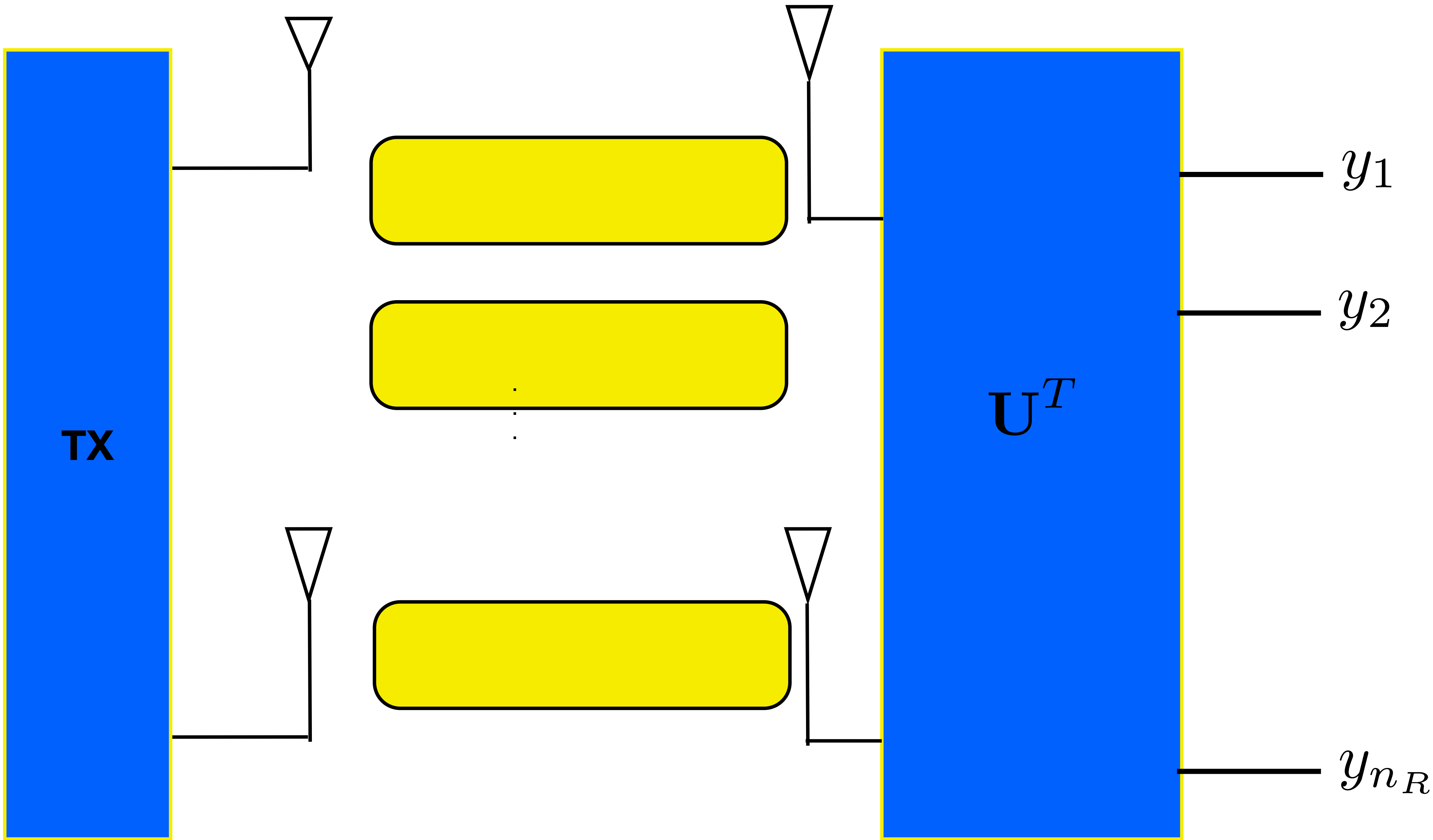
$$\mathbf{y} = \mathbf{U}^T \mathbf{r}$$

or

$$\mathbf{y} = \mathbf{U}^T \mathbf{H}\mathbf{x} + \mathbf{U}\mathbf{n}$$

$$= \mathbf{U}^T \mathbf{Q}\mathbf{R}\mathbf{x} + \mathbf{U}\mathbf{n}$$

$$= \mathbf{R}\mathbf{x} + \mathbf{n}'$$


TX
$\mathbf{U}^T$
y_1
y_2
y_{n_R}

$$
\begin{aligned}
\mathbf{y} &= \mathbf{U}^T\mathbf{r} = \mathbf{U}^T(\mathbf{H}\mathbf{x} + \mathbf{n}) \\
&= \mathbf{U}^T(\mathbf{U}\mathbf{R}\mathbf{x} + \mathbf{n})
\end{aligned}
$$

$$
= \begin{bmatrix} R_{1,1} & R_{1,2} & \cdots & R_{1,n_T} \\ 0 & R_{2,2} & \cdots & R_{2,n_T} \\ 0 & 0 & \cdots & R_{2,n_T} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & R_{n_T,n_T} \end{bmatrix} \left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n_T} \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_{n_R} \end{bmatrix} \right)
$$

$$
\begin{aligned}
y_{n_T} &= R_{n_T,n_T} x_{n_T} + n'_{n_T} \\
y_{n_T-1} &= R_{n_T-1,n_T-1} x_{n_T-1} + R_{n_T-1,n_T} x_{n_T} + n'_{n_T-1} \\
&\vdots \\
y_1 &= R_{1,1} x_1 + R_{1,2} x_2 + \cdots + R_{1,n_T} x_{n_T} + n'_1
\end{aligned}
$$

or simply

$$
y_k = R_{k,k} x_k + \sum_{j=k+1}^{n_T} R_{k,j} x_j + n'_k \quad k = 1, 2, \ldots, n_T
$$

Self-Interference

Decision statistics

$$\hat{x}_k = q\left(\frac{y_k - \sum_{j=k+1}^{n_T} R_{k,j}x_j}{R_{k,k}}\right), \qquad i = 1, 2, \ldots, n_T$$

where $q(\cdot)$ is the hard decision operation.

- Example for 3 by 3 antennas with the channel matrix given as

$$
\begin{aligned}
y_1 &= R_{1,1}x_1 + R_{1,2}x_2 + R_{1,3}x_3 + n_1' \\
y_2 &= R_{2,2}x_2 + R_{2,3}x_3 + n_3' \\
y_3 &= R_{33}x_3 + n_3'
\end{aligned}
$$

$$\hat{x}_3 = q\left(\frac{y_3}{R_{3,3}}\right)$$

$$\hat{x}_2 = q\left(\frac{y_2 - R_{2,3}\hat{x}_3}{R_{2,2}}\right)$$

$$\hat{x}_1 = q\left(\frac{y_1 - R_{1,3}\hat{x}_3 - R_{1,2}\hat{x}_2}{R_{1,1}}\right.$$