Communication Systems II

[KECE322_01]

<2012-2nd Semester>

Lecture #9

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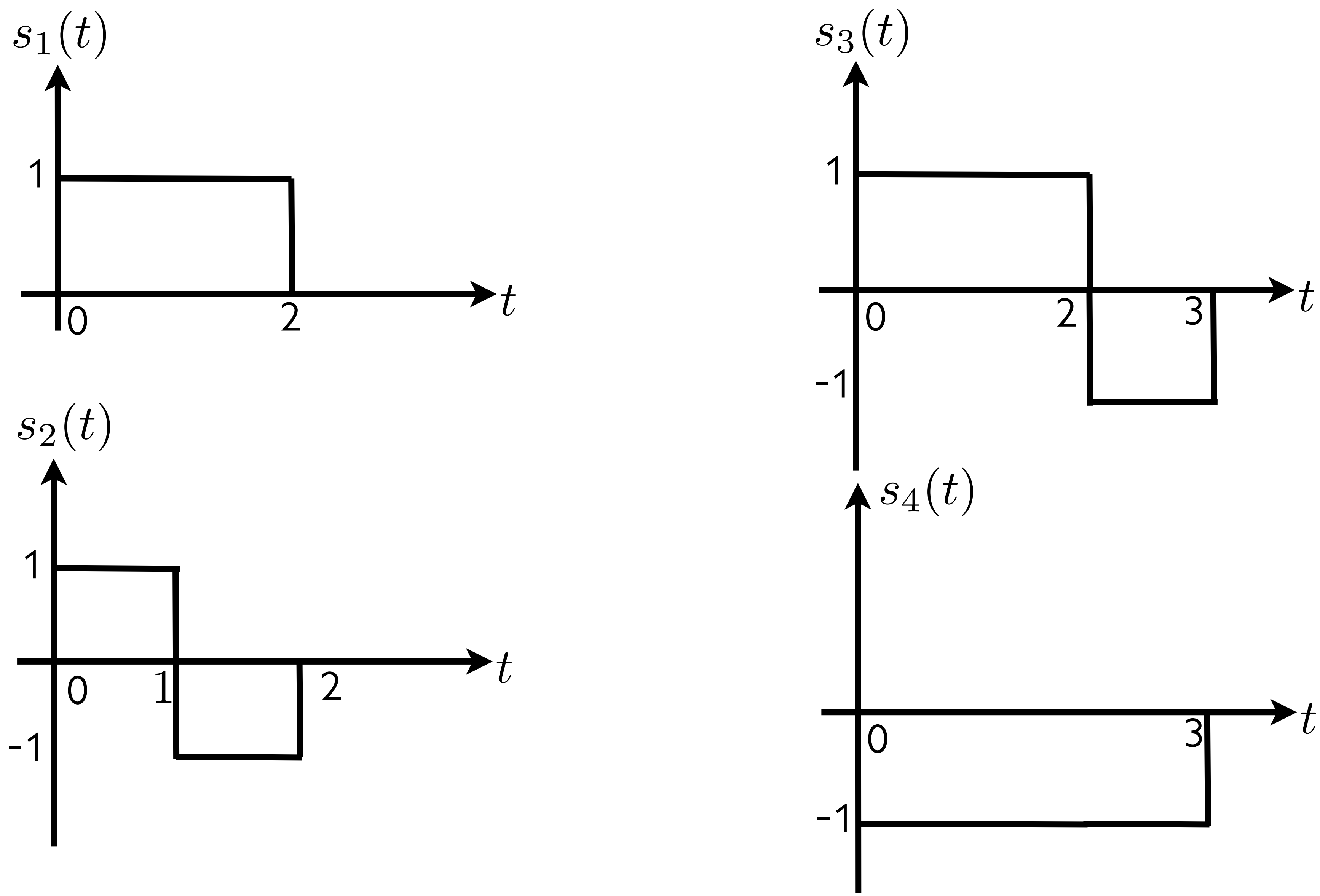
Outline

- Gram-Schmidt procedure
- Optimum receiver over AWGN

Example of Gram-Schmidt Procedure

- Find the orthonormal functions for the set of four waveforms $\{s_k(t)\}_{k=1}^4$



Gram-Schmidt procedure

- The waveform $s_1(t)$ has energy $\mathcal{E}_1 = 2$, so that

$$\psi_1(t) = \sqrt{\frac{1}{2}} s_1(t)$$

- We observe that $c_{12} = 0$. Hence, $s_2(t)$ are orthogonal to $\psi_1(t)$. Therefore,

$$\phi_2(t) = \frac{s_2(t)}{\sqrt{\mathcal{E}_2}}$$

- To obtain $\phi_3(t)$, we compute c_{13} and c_{23}, which are $c_{13} = \sqrt{2}$ and $c_{23} = 0$. Thus,

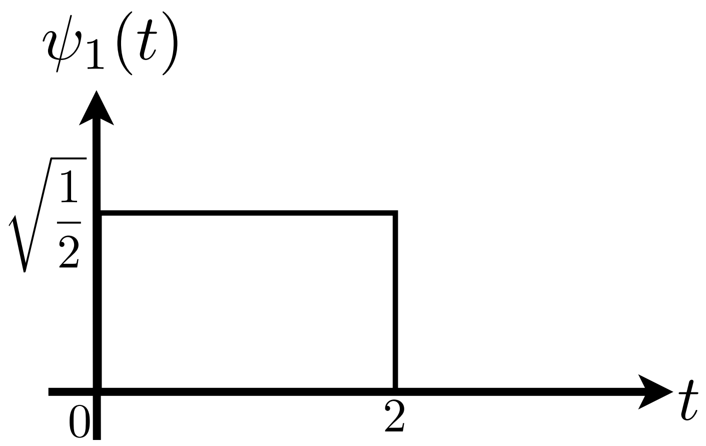
$$d_3(t) = s_3(t) - \sqrt{2}\psi_1(t) = \begin{cases} -1, & (2 \leq t \leq 3) \\ 0, & (\text{otherwise}). \end{cases}$$

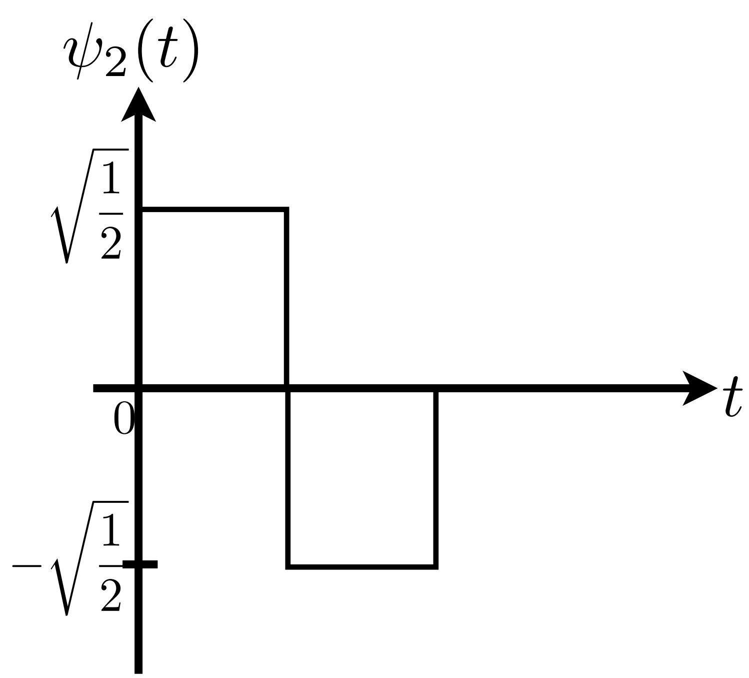
  - Since $d_3(t)$ has unit energy, it follows that $\psi_3(t) = d_3(t)$.

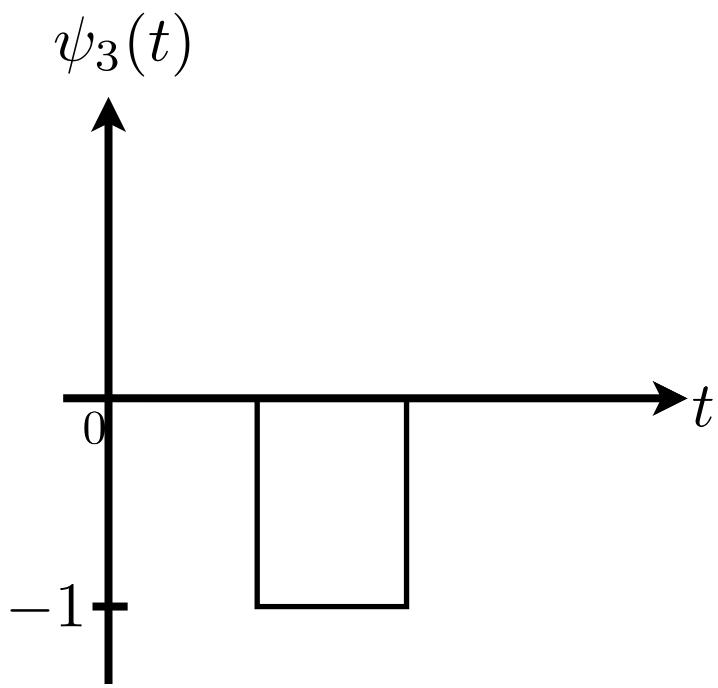
- In determining $\psi_4(t)$, we find that $c_{14} = -\sqrt{2}$, $c_{24} = 0$, and $c_{34} = 1$. Hence,

$$d_4(t) = s_4(t) + \sqrt{2}\phi_1(t) - \psi(t) = 0$$

- Consequently, $s_4(t)$ is a linear combination of $\psi_1(t)$ and $\psi_3(t)$, hence, $\psi_4(t) = 0$.







Geometrical Representation of Signals

- Once we have constructed the set of orthogonal waveforms $\{\psi_n(t)\}_{n=1}^{N}$, we can express the signals $\{s_m(t)\}_{m=1}^{M}$ as exact combinations of the $\{\psi_n(t)\}_{n=1}^{N}$.
  - Hence, we may write

Dimension

$$s_m(t) = \sum_{n=1}^{N} s_{mn}\psi_n(t), \quad m = 1, 2, \ldots, M.$$

where $$s_{mn} = \int_{-\infty}^{\infty} s_m(t)\psi_n(t)\, dt.$$

- Signal energy

$$\mathcal{E}_m = \int_{-\infty}^{\infty} s_m^2(t)\, dt = \sum_{n=1}^{N} s_{mn}^2.$$

- Vector representation

  - For $s_m(t) = \sum_{n=1}^{N} s_{mn}\phi_n(t)$, the vector representation of $s_m(t)$ is defined as

$$\mathbf{s}_m = \left[s_{m1} \;\; s_{m2} \;\; \cdots \;\; s_{mN}\right]$$

- Inner product of two signals

$$\mathbf{s}_m \cdot \mathbf{s}_n = \int_{-\infty}^{\infty} s_m(t)s_n(t)\,dt \;\; = \sum_{k=1}^{N} s_{mk}s_{nk}$$

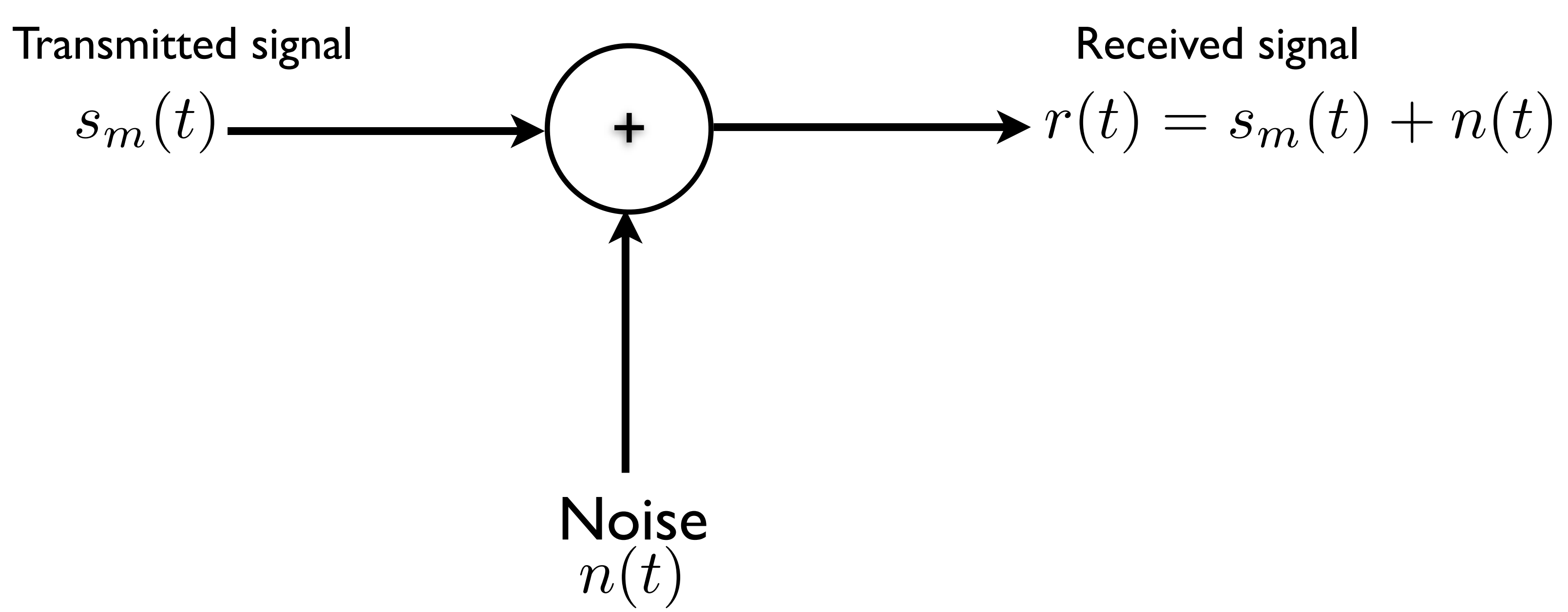
Additive White Gaussian Noise Channel

- Received signal in a signal interval of duration T_b over AWGN channel

$$r(t) = s_m(t) + n(t), \quad m = 1, 2,$$

  - $n(t)$ denotes the sample function of the additive white Gaussian noise (AWGN) process with the power spectral density $S_n(f) = N_0/2 \,\mathrm{W/Hz}.$

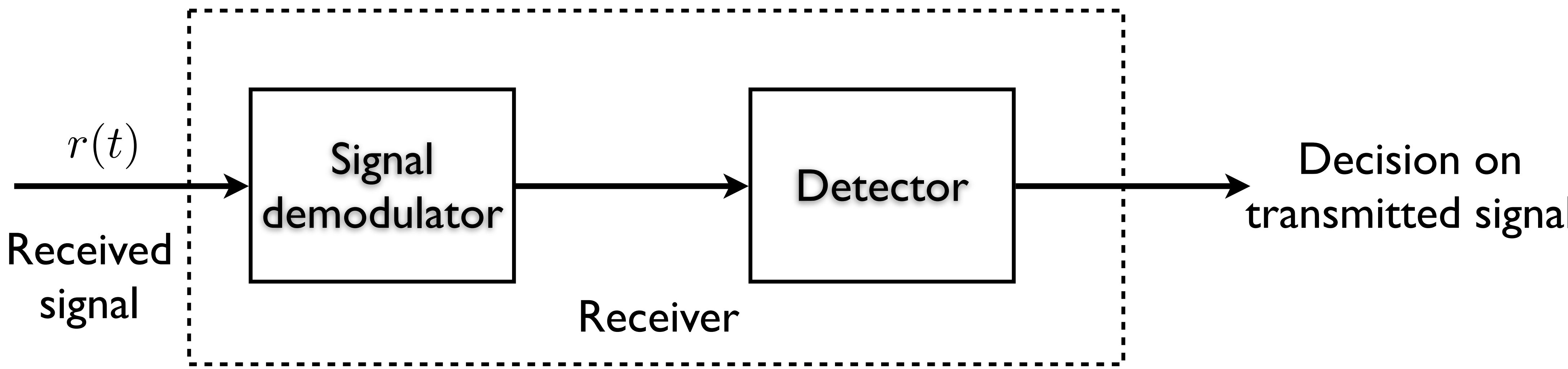
- Block diagram of AWGN channel



Optimum Receiver over AWGN

- Based on the observation of $r(t)$ over the signal interval, we wish to design a receiver that is optimum *in the sense that it minimizes the probability of making an error.*

- Receiver structure



- Two types of signal demodulator
  - Correlation-type demodulator
  - Matched filter-type demodulator

Correlation-Type Demodulator for Binary Antipodal Signals

- Signal waveform

$$s_m(t) = s_m \psi(t), \;\; m = 1, 2$$

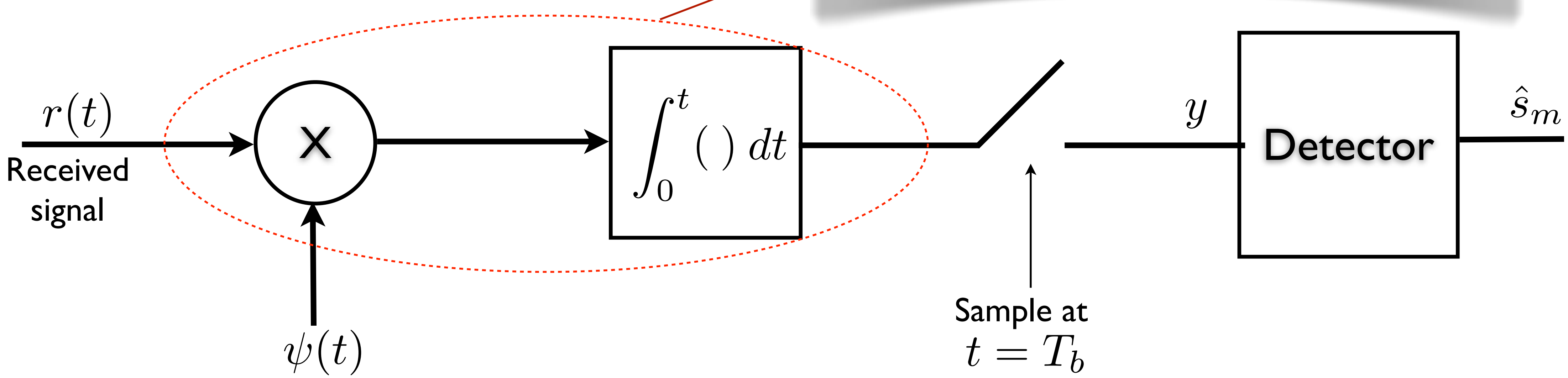
  - where $\psi(t)$ is the unit energy rectangular pulse and $s_1 = \sqrt{\mathcal{E}_b}, \;\; s_2 = -\sqrt{\mathcal{E}_b}$.

- Received signal

$$r(t) = s_m \psi(t) + n(t), \;\; 0 \leq t \leq T_b, \;\; m = 1, 2.$$

- Correlation-type demodulator

$r(t)$ is cross-correlated with $\psi(t)$.



- Output of cross-correlation operation

$$\begin{aligned} y(t) &= \int_0^t r(\tau)\psi(\tau)\, d\tau \\ &= \int_0^t [s_m\psi(\tau) + n(\tau)]\psi(\tau)\, d\tau \\ &= s_m \int_0^t \psi^2(\tau)\, d\tau + \int_0^t n(t)\psi(\tau)\, d\tau. \end{aligned}$$

- Sampling the output of the correlator at $t = T_b$

$$y(T_b) = s_m + n$$

desired signal term (s_m)   noise term (n)

- where

$$n = \int_0^{T_b} \psi(\tau)n(\tau)\, d\tau$$

Noise term

$$n = \int_0^{T_b} \psi(\tau)n(\tau)\, d\tau$$

- n is Gaussian random variable.

- Mean

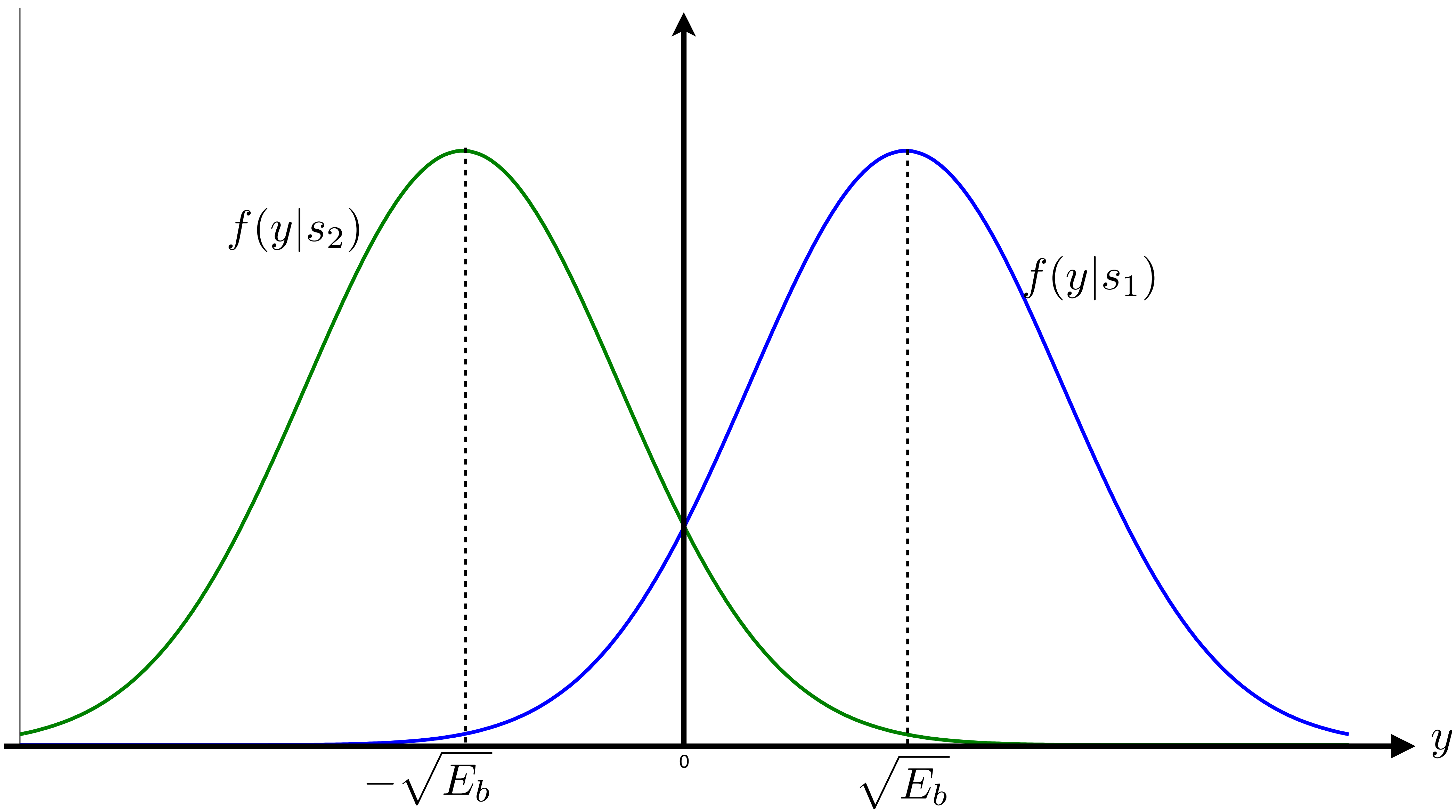
$$E[n] = E\left[\int_0^{T_b} \psi(\tau)n(\tau)\, d\tau\right] = \int_0^{T_b} \psi(\tau)E[n(\tau)]\, d\tau = 0$$

- Variance

$$\begin{aligned}
\sigma_n^2 &= E[n^2] = \int_0^{T_b}\int_0^{T_b} E[n(t)n(\tau)]\psi(t)\psi(\tau)\, dtd\tau \\
&= \int_0^{T_b}\int_0^{T_b} \frac{N_0}{2}\delta(t-\tau)\psi(t)\psi(\tau)\, dtd\tau \\
&= \frac{N_0}{2}\int_0^{T_b} \psi^2(t)\, dt = \frac{N_0}{2}.
\end{aligned}$$

- Conditional PDF given s_m

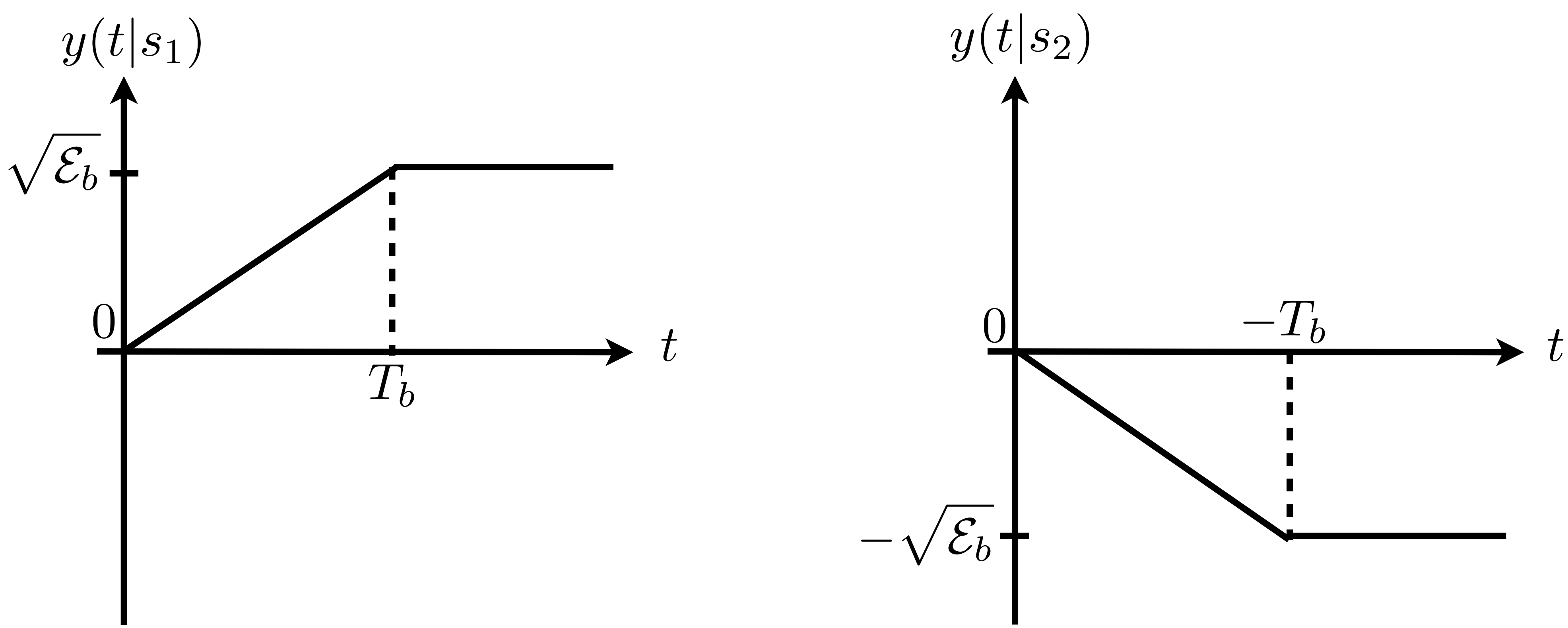
$$f(y|s_m) = \frac{1}{\sqrt{\pi N_0}} e^{-(y-s_m)^2/N_0}, \quad m = 1, 2.$$



- Noise-free output of the correlator for the rectangular pulse $\psi(t)$

With $n(t) = 0$, the signal waveform at the output of the correlator is

$$y(t) = \int_0^t s_m \psi^2(\tau)\, d\tau = s_m \int_0^t \psi^2(t)\, d\tau$$



- Note that the maximum signal at the output of the correlator occurs at $t = T_b$.

- We also observe that the correlator must be reset to zero at the end of each bit interval T_b, so that it can be used in the demodulator of the received signal in the next signal interval. Such an integrator is called an integrate-and-dump filer.

Correlation-Type Demodulator for Binary Orthogonal Signals

- Signal waveform

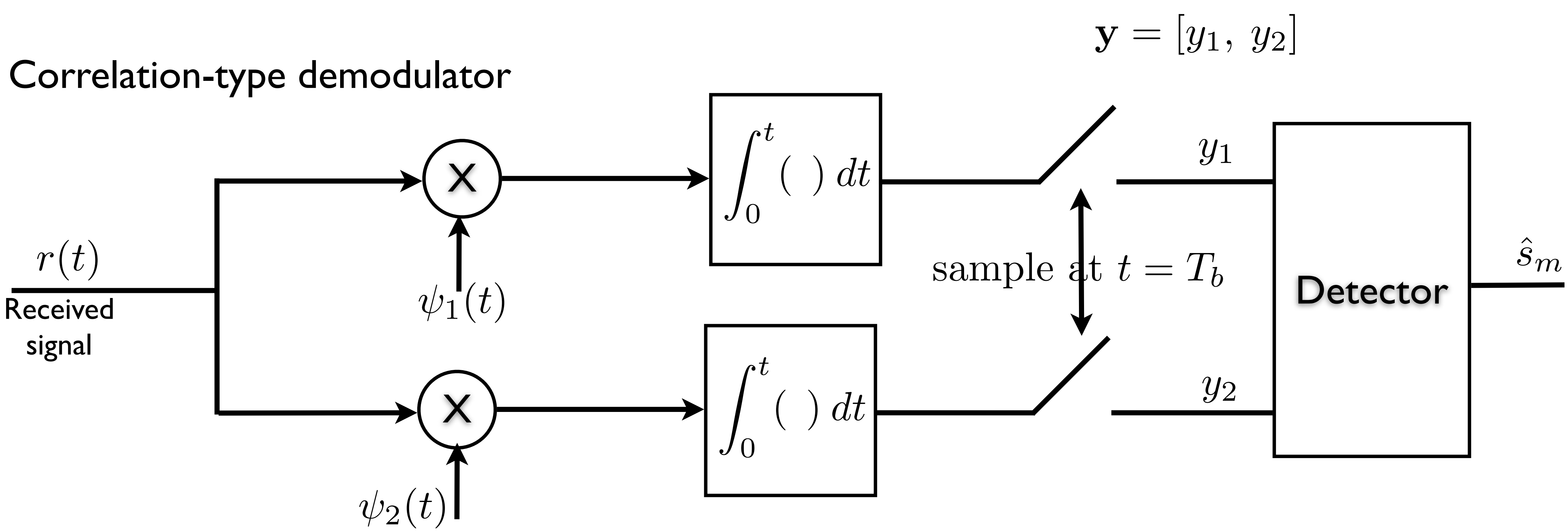
$$r(t) = s_m(t) + n(t), \quad 0 \le t \le T_b, \quad m = 1, 2.$$

where $s_1(t) = \sqrt{\mathcal{E}_b}\psi_1(t)$, and $s_2(t) = \sqrt{\mathcal{E}_b}\psi_2(t)$

  - Note that in vector form, the transmit signals are

$$\mathbf{s}_1 = [\sqrt{\mathcal{E}_b},\ 0], \text{ and } \mathbf{s}_2 = [0,\ \sqrt{\mathcal{E}_b}]$$

- Correlation-type demodulator



- Correlator output waveforms

$$y_m(t) = \int_0^t r(\tau)\phi_m(\tau)\, d\tau, \;\; m = 1, 2.$$

- Sampled signal at $t = T_b$

$$y_m = y_m(T_b) = \int_0^{T_b} r(\tau)\phi_m(\tau)\, d\tau, \;\; m = 1, 2.$$

  - For $s_1(t) = s_{11}\phi_1(t)$, so that $r(t) = s_{11}\psi_1(t) + n(t)$.

$$y_1 = \int_0^{T_b} [s_{11}\psi_1(\tau) + n(\tau)]\psi_1(\tau)\, d\tau = s_{11} + n_1 = \sqrt{E_b} + n_1$$

$$y_2 = \int_0^{T_b} [s_{11}\psi_1(t) + n(t)]\psi_2(t)\, dt = n_2$$

where

$$n_1 = \int_0^{T_b} n(\tau)\psi_1(\tau) d\tau$$

$$n_2 = \int_0^{T_b} n(\tau)\psi_2(\tau) d\tau$$

- Sampled output in vector form if $s_1(t)$ is transmitted:

$$\mathbf{y} = [y_1,\ y_2] = [\sqrt{\mathcal{E}_b} + n_1,\ n_2]$$

- Sampled output in vector form if $s_2(t)$ is transmitted:

$$\mathbf{y} = [y_1,\ y_2] = [n_1, \sqrt{\mathcal{E}_b} + n_2]$$

Statistical characteristic of the observed signal vector $\mathbf{y}$

- n_1 and n_2 are zero-mean Gaussian random variable with variance $\sigma^2 = N_0/2$.

$$n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$$

- Correlation between n_1 and n_2

$$\begin{aligned} E[n_1 n_2] &= \int_0^{T_b}\int_0^{T_b} E[n(t)n(\tau)\psi_1(t)\psi_2(\tau)\, dt\, d\tau \\ &= \int_0^{T_b}\int_0^{T_b} \frac{N_0}{2}\delta(t-\tau)\psi_1(t)\psi_2(\tau)\, dt\, d\tau \\ &= \frac{N_0}{2}\int_0^{T_b} \psi_1(t)\psi_2(\tau)\, dt\, d\tau = 0. \end{aligned}$$

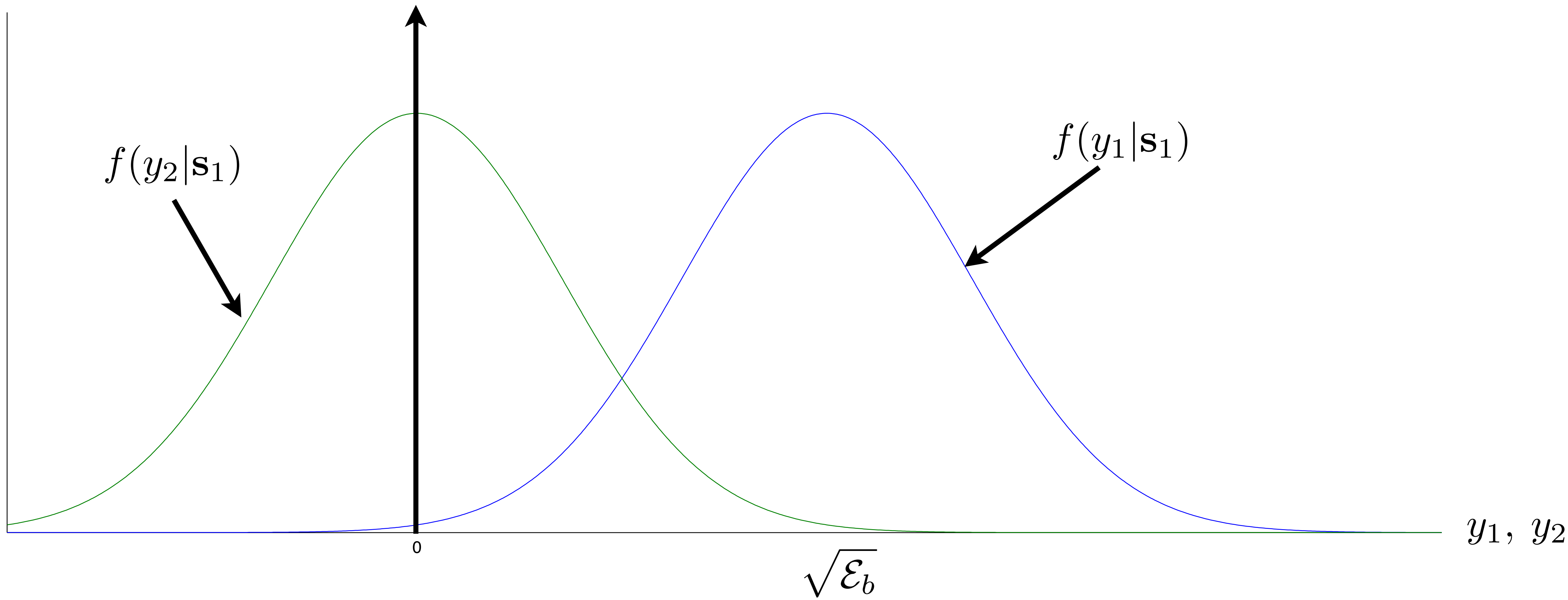
- Conditional joint PDF

$$\begin{aligned} f(y_1, y_2|\mathbf{s}_1) &= \left(\frac{1}{\sqrt{\pi N_0}}\right)^2 \exp\left[-\frac{(y_1-\sqrt{\mathcal{E}_b})^2 + y_2^2}{N_0}\right] \\ f(y_1, y_2|\mathbf{s}_2) &= \left(\frac{1}{\sqrt{\pi N_0}}\right)^2 \exp\left[-\frac{y_1^2 + (y_2-\sqrt{\mathcal{E}_b})^2}{N_0}\right] \end{aligned}$$

$\Rightarrow$

$$f(y_1, y_2|\mathbf{s}_m) = f(y_1|\mathbf{s}_m)f(y_2|\mathbf{s}_m)$$

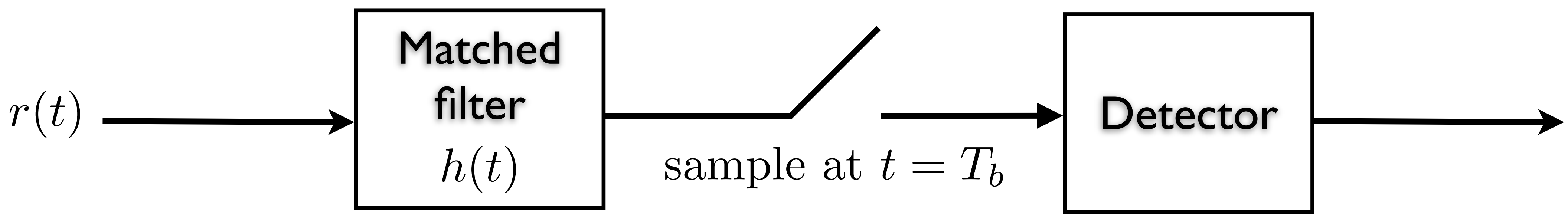
- Conditional PDF when $s_1(t)$ is transmitted.



Matched Filter Type Demodulator

- Binary antipodal signals

$$r(t) = s_m\psi(t) + n(t), \quad 0 \le t \le T_b, \quad m = 1, 2$$



  - Impulse response of matched filter

$$h(t) = \psi(T_b - t), \quad 0 \le t \le T_b$$

  - Filter output

$$y(t) = \int_0^t r(\tau)h(t-\tau)\,d\tau$$

- Sampling at time $t = T_b$

$$y(T_b) = \int_0^{T_b} r(\tau)h(T_b - \tau)\, d\tau$$

Since $h(T_b - \tau) = \psi(\tau)$

the sampled output signal is

$$\begin{aligned} y(T_b) &= \int_0^{T_b} [s_m\psi(\tau) + n(\tau)]\psi(\tau)\, d\tau \\ &= s_m + n \end{aligned}$$

where

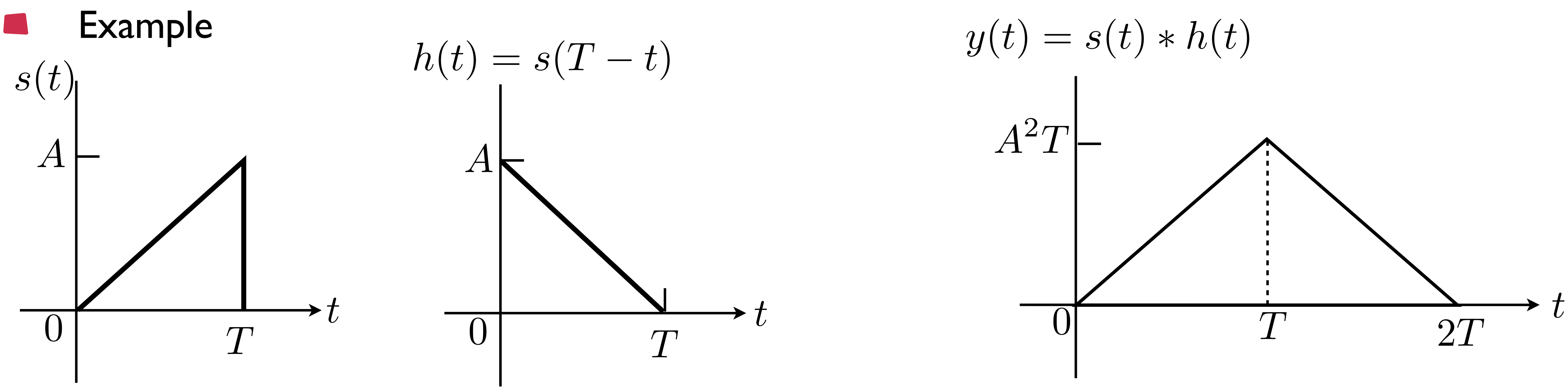
$$n = \int_0^{T_b} n(\tau)\psi(\tau)\, d\tau$$

✦ The sampled output is exactly the same as the output obtained with a cross-correlator.

Matched Filter

- Definition:
  - A filter whose impulse response $h(t) = s(T-t)$, where $s(t)$ is assumed to be confined to the time interval $0 \le t \le T$.

- Example



Binary Orthogonal Signals with Matched Filter

- Binary orthogonal signal waveforms

$$r(t) = s_m(t) + n(t), \;\; 0 \leq t \leq T_b, \;\; m = 1, 2$$

where

$$< s_1(t), s_2(t) >= \int_0^{T_b} s_1(t)s_2(t)\, dt = 0$$

- Consider matched filters with impulse response given as

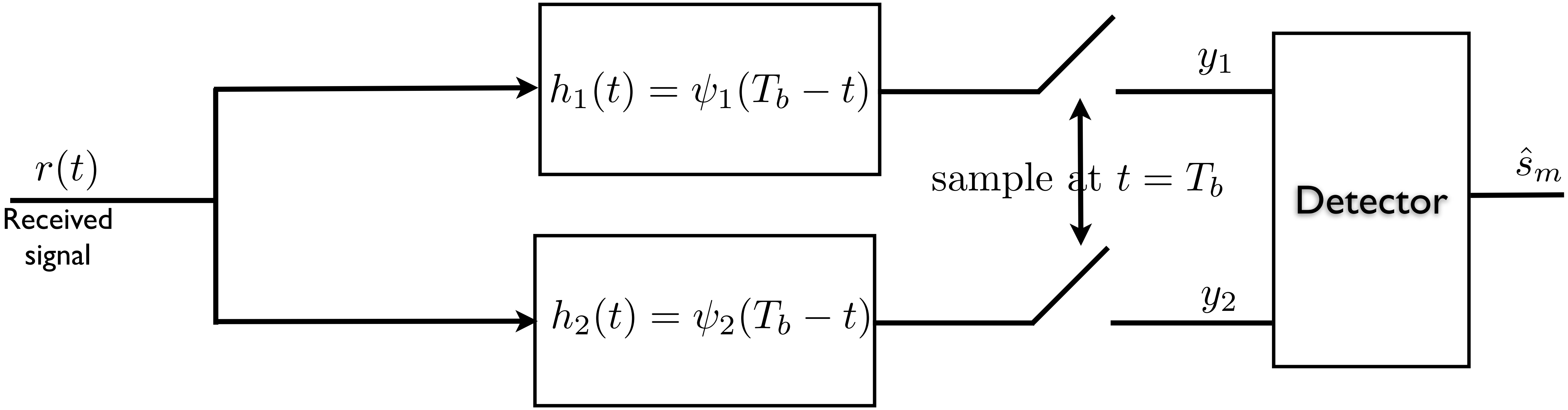
$$\begin{aligned} h_1(t) &= \psi_1(T_b - t), \;\; 0 \leq t \leq T_b \\ h_2(t) &= \psi_2(T_b - t), \;\; 0 \leq t \leq T_b \end{aligned}$$

- Output at the matched filter

$$y_m(t) = \int_0^t r(\tau)h_m(t-\tau)\, d\tau, \;\; m = 1, 2.$$

Sampled output

$$
\begin{aligned}
y_m = y_m(T_b) &= \int_0^{T_b} r(\tau) h_m(T_b - \tau)\, d\tau \\
&= \int_0^{T_b} r(\tau) \psi_m(\tau)\, d\tau, \quad m = 1, 2
\end{aligned}
$$



When $s_1(t)$ was transmitted,

$$
\begin{aligned}
y_1 &= s_{11} + n_1 \\
y_2 &= n_2
\end{aligned}
$$