Numerical Analysis MTH614

Spring 2012, Korea University

Finite Differences for the Heat Equation 1

The heat equation

Many physical processes are governed by partial differential equations. One such phenomenon is heat energy spreading out into space.

$$\partial_t u = \triangle u,\ \mathbf{x} \in \mathbf{R}^n,\ t > 0. \quad (1)$$

Initial and boundary conditions

We consider one dimensional problem such as

$$\partial_t u(x,t) = u_{xx}(x,t),\ 0 < x < 1,\ t > 0. \quad (2)$$

Its boundary and initial condition are given

$$u(0,t) = u(1,t) = 0,\ u(x,0) = \sin(\pi x).$$

Then it has a unique solution $u(x,t) = \sin(\pi x) e^{-\pi t}$. (3)

Finite difference schemes

Generally, it can be represented as

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} + O(h).$$

where h is called the step size.

It is termed a "forward" difference.

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} + O(h).$$

This is a "backward" difference.

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{x_{i+1} - x_{i-1}} + O(h^2)$$

The equation is a "centered" difference.

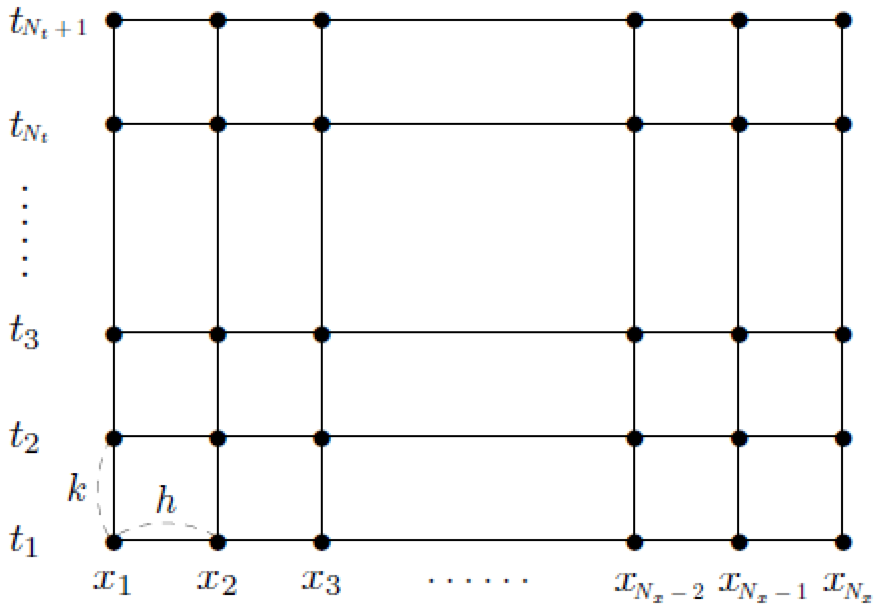
Use these to verify the derivatives such as

$$u(x) = \sin x \text{ at } x = \pi/4.$$

```
%%%%%%%%%%%%%%%%%%%%%%%%%%% difference.m %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear; clc; x=pi/4;
fprintf('    h         forward      backward     central \n')
for k=1:10
h=1/2^k;
forw=(sin(x+h)-sin(x))/h;
back=(sin(x)-sin(x-h))/h;
cent=(sin(x+h)-sin(x-h))/(2*h);
fprintf('%.6f     %.6f     %.6f     %.6f \n',h,forw,back,cent)
end
```

h	forward	backward	central
0.500000	0.504886	0.851135	0.678010
0.250000	0.611835	0.787693	0.699764
0.125000	0.661130	0.749403	0.705267
0.062500	0.684557	0.728736	0.706647
0.031250	0.695944	0.718039	0.706992
0.015625	0.701554	0.712602	0.707078
0.007813	0.704337	0.709862	0.707100
0.003906	0.705724	0.708486	0.707105
0.001953	0.706416	0.707797	0.707106
0.000977	0.706761	0.707452	0.707107

Mesh



$$h = (b-a)/(N_x - 1),\ k = T/N_t,\ t_n = (n-1)\cdot k,\ x_i = a + (i-1)\cdot h, \quad (1)$$

$$\begin{aligned} a = x_1 < x_2 < \cdots < x_{N_x-1} < x_{N_x} = b \\ 0 = t_1 < t_2 < \cdots < t_{N_t} < t_{N_t+1} = T \end{aligned} \quad (2)$$

$u(x_i, t_n)$ is thereafter denoted as u_i^n.

Explicit finite difference

Consider the one dimensional heat equation,

$$\partial_t u(x,t) = u_{xx}(x,t),\ 0 < x < 1,\ t > 0. \qquad (1)$$

Then, the explicit finite difference scheme based on centered differences in space and a forward difference in time yields that

$$\frac{u_i^{n+1} - u_i^n}{k} + O(k) = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} + O(h^2) \qquad (2)$$

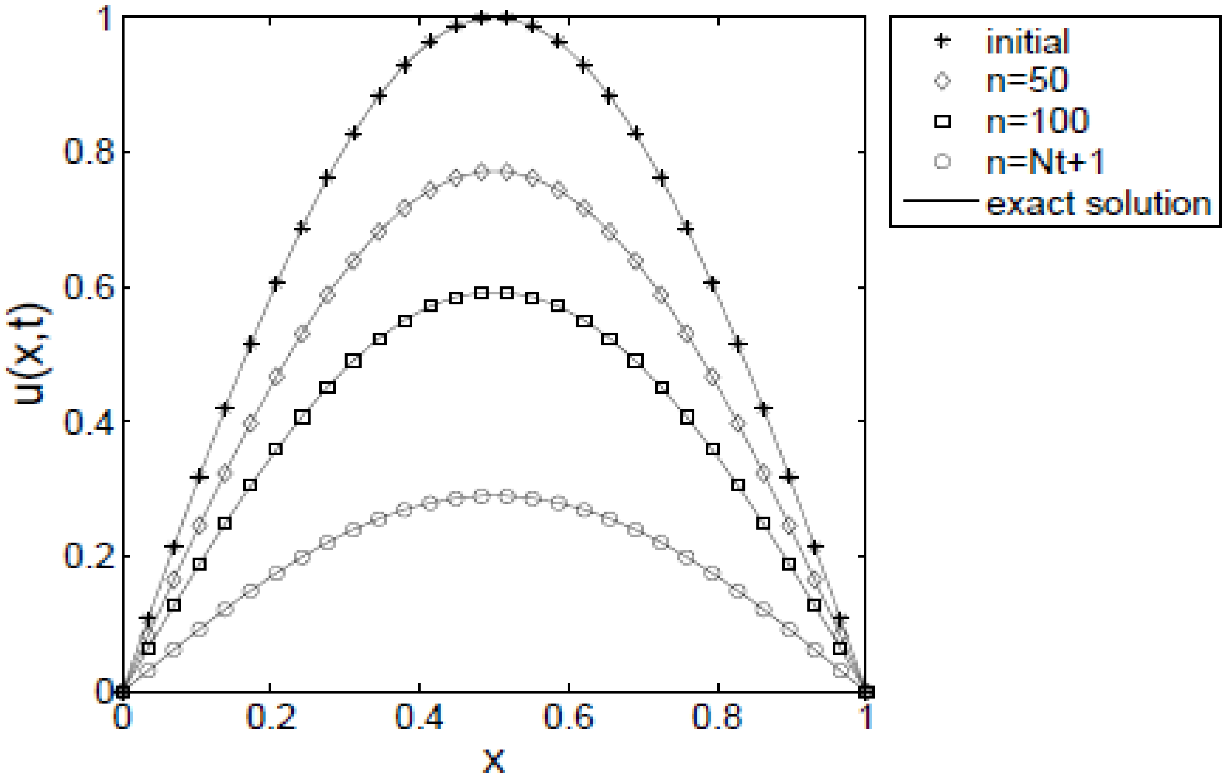
$$\text{for}\ \ i = 2, \ldots, N_x - 1\ \ \text{and}\ \ n = 1, 2, \ldots, N_t.$$

Then, the explicit finite difference scheme based on centered differences in space and a forward difference in time yields that

$$u_i^{n+1} = u_i^n + \alpha(u_{i+1}^n - 2u_i^n + u_{i-1}^n),\ \alpha = k/h^2 \qquad (3)$$

We let $\alpha = 0.45,\ N_x = 30,\ \text{subject to}\ u_1^n = u_{N_x}^n = 0.$

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%% heatex.m %%%%%%%%%%%%%%%%%%%%%%%%%%%
clf; clear; clc; alpha=0.45; Nx=30; x=linspace(0,1,Nx);
h=x(2)-x(1); k=alpha*h^2; T=0.125; Nt=round(T/k);
u(1:Nx,1:Nt+1)=0; u(:,1)=sin(pi*x); exu=u;
for n=1:Nt
    for i=2:Nx-1
        u(i,n+1) = u(i,n)+alpha*(u(i-1,n)-2*u(i,n)+u(i+1,n));
      exu(i,n+1) = sin(pi*x(i))*exp(-pi^2*(k*n));
    end
end
plot(x,u(:,1),'k*',x,u(:,50),'kd',x,u(:,100),'ks',...
     x,u(:,Nt+1),'ko'); hold
plot(x,exu(:,1),'k',x,exu(:,50),'k',x,exu(:,100),'k',...
     x,exu(:,Nt+1),'k')
legend('initial','n=50','n=100','n=Nt+1','exact solution',-1)
xlabel('x','FontSize',20); ylabel('u(x,t)','FontSize',20)
```



1
0.8
0.6
0.4
0.2
0
u(x,t)
0
0.2
0.4
0.6
0.8
1
x
initial
n=50
n=100
n=Nt+1
exact solution

Explicit stability (von Neumann)

With the help of the von Neumann analysis, we examine stability of finite difference schemes. Let us begin by introducing finite Fourier series

$$u_k^n = e^{i\beta kh}\xi^n. \quad (1)$$

We illustrate the method by considering the heat equation. If we rewrite the heat equation as explicitly,

$$u_i^{n+1} = u_i^n + \alpha(u_{i+1}^n - 2u_i^n + u_{i-1}^n), \quad (2)$$

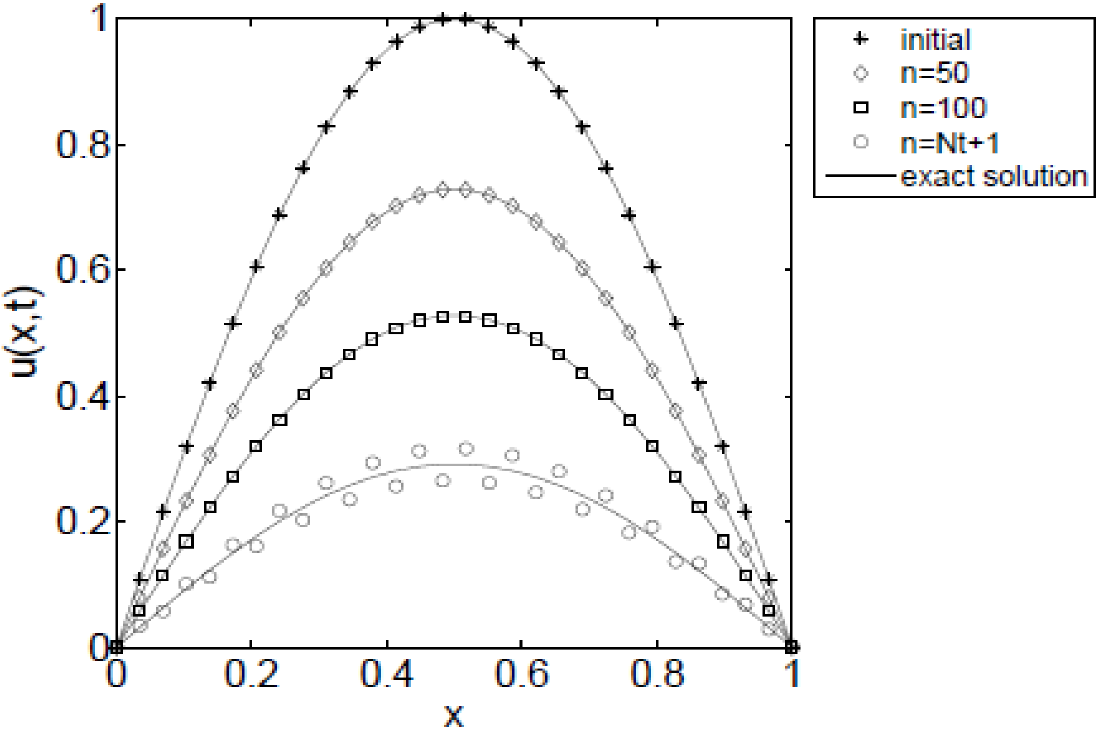
Using the Fourier series for u_i^n

$$\begin{aligned} e^{i\beta kh}\xi^{n+1} &= \alpha e^{i\beta(k-1)h}\xi^n + (1-2\alpha)e^{i\beta kh}\xi^n + \alpha e^{i\beta(k+1)h}\xi^n, \\ \xi &= \alpha e^{-i\beta h} + (1-2\alpha) + \alpha e^{i\beta h} = 1 - 4\alpha\sin^2\frac{\beta h}{2}. \end{aligned} \quad (3)$$

To stabilize the scheme, we have to contain this term in

$$0 \le \alpha\sin^2\frac{\beta h}{2} \le \frac{1}{2}.$$

That is, $0 < \alpha \le \dfrac{1}{2}.$



1
0.8
0.6
0.4
0.2
0
u(x,t)
0
0.2
0.4
0.6
0.8
1
x
initial
n=50
n=100
n=Nt+1
exact solution

Implicit finite difference

We now examine an implicit scheme. Consider the one-dimensional heat equation subject to homogeneous Dirichlet boundary conditions

$$\partial_t u(x,t) = u_{xx}(x,t),\ 0 < x < 1,\ t > 0,$$

$$u(0,t) = u(1,t) = 0,\ u(x,0) = \sin(\pi x).$$

Thus, this time we approximate the solution to a value of u using the implicit finite difference scheme consisting of a backward in time and centered difference in space. Then we have

$$\frac{u_i^{n+1} - u_i^n}{k} = \frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{h^2}. \quad (1)$$

Rewrite this,

$$-\alpha u_{i-1}^{n+1} + (1+2\alpha)u_i^{n+1} - \alpha u_{i+1}^{n+1} = u_i^n,\ \alpha = \frac{k}{h^2},\ i = 2, \cdots, N_x - 1\ . \quad (2)$$

To solve this, we use the linear system equations:

$$\begin{pmatrix} 1+2\alpha & -\alpha & 0 & \dots & 0 \\ -\alpha & 1+2\alpha & -\alpha & & 0 \\ 0 & -\alpha & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -\alpha \\ 0 & 0 & & -\alpha & 1+2\alpha \end{pmatrix} \begin{pmatrix} u_2^{n+1} \\ u_3^{n+1} \\ \vdots \\ \vdots \\ u_{N_x-1}^{n+1} \end{pmatrix} = \begin{pmatrix} \alpha u_1^{n+1} + u_2^n \\ u_3^n \\ \vdots \\ \vdots \\ u_{N_x-1}^n + \alpha u_{N_x}^{n+1} \end{pmatrix} = \begin{pmatrix} b_2^n \\ b_3^n \\ \vdots \\ \vdots \\ b_{N_x-1}^n \end{pmatrix}. \qquad (1)$$

We consider the equation (1) of $\mathbf{A}\boldsymbol{u}^{n+1} = \boldsymbol{b}^n$, where (2)

$$\boldsymbol{u}^{n+1} = (u_2^{n+1}, \cdots, u_{N_x-1}^{n+1})^T, \quad \boldsymbol{b}^n = \boldsymbol{u}^n + \alpha(u_1^{n+1}, 0, \cdots, 0, u_{N_x}^{n+1})^T. \qquad (3)$$

Since the A is invertible,

$$\boldsymbol{u}^{n+1} = \mathbf{A}^{-1}\boldsymbol{b}^n. \qquad (4)$$

Implicit stability (von Neumann)

Let us apply the von Neumann analysis to investigate the stability of the implicit scheme. We see the effect of the scheme on a complex exponential

$$u_k^n = e^{i\beta kh}\xi^n \tag{1}$$

Substituting (1) into the implicit scheme (2)

$$-\alpha u_{i-1}^{n+1} + (1+2\alpha)u_i^{n+1} - \alpha u_{i+1}^{n+1} = u_i^n, \ \alpha = \frac{k}{h^2}, \ i = 2, \cdots, N_x - 1 \tag{2}$$

We solve for the ξ

$$\begin{aligned} -\alpha e^{i\beta(k-1)h}\xi^{n+1} + (1+2\alpha)e^{i\beta kh}\xi^{n+1} - \alpha e^{i\beta(k+1)h}\xi^{n+1} &= e^{i\beta kh}\xi^n, \\ -\alpha e^{-i\beta h}\xi + (1+2\alpha)\xi - \alpha e^{i\beta h}\xi &= 1, \\ (2\alpha(1-\cos(\beta h))+1)\xi &= 1. \end{aligned} \tag{3}$$

That is $\xi = \dfrac{1}{4\alpha\sin^2(\beta h/2)+1}.$ (4)

Since $\alpha > 0, \text{ for all } \beta,$ we have $\frac{1}{4\alpha+1} \leq \xi \leq 1.$ (5)