Chapter 4. Symmetry in QM (09/05/2011)

Symmetries in Classical Mechanics (1)

$L(q, \dot{q}) \to L'(q+\delta q$ (Graduate)

$$\delta L = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}$$

$$= \frac{\partial L}{\partial q}\delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q$$

$$= \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right)$$

$$\int \delta L\, dt = \int \underbrace{[\qquad]}_{\substack{\parallel \\ 0}} \delta q\, dt + \cancel{\frac{\partial L}{\partial \dot{q}}\delta q}\Big|_{t=t_1}^{t=t_2}$$

$\leftarrow$ E-L equation

if $\frac{\partial L}{\partial q} = 0$ then $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$

cyclic coordinate $\quad p = \frac{\partial L}{\partial \dot{q}}$ is conserved.

Symmetry in QM.

$\mathcal{S}$: symmetry operator.

$$\mathcal{S} = 1 - \frac{i}{\hbar}\varepsilon G.$$

$\mathcal{S}^{\dagger} H \mathcal{S} = H.$ invariant.

$$H\mathcal{S} = \mathcal{S}H \rightarrow [H, \mathcal{S}] = 0.$$

$$\Leftrightarrow [H, G] = 0$$

$$\rightarrow \frac{dG}{dt} = 0.$$

Heisenberg eq. of motion.

Degeneracy

$[H, \mathcal{S}] = 0.$

$$H\mathcal{S}|n\rangle = \mathcal{S}H|n\rangle = \mathcal{S}E_n|n\rangle$$

$$= E_n \mathcal{S}|n\rangle$$

$\therefore$ $|n\rangle$ & $\mathcal{S}|n\rangle$ are degenerate.

if $\langle n|\mathcal{S}|n\rangle \doteq 0$

then

$[J, H] = 0$

Example. $[\mathcal{D}(R), H] = 0 \rightarrow [J^2, H] = 0$

4.2 Discrete symmetries

Parity $|\alpha\rangle \to \pi|\alpha\rangle$ $(\vec{x} \to -\vec{x})$

$$\langle\alpha|\pi^{+}\hat{\vec{x}}\pi|\alpha\rangle = -\langle\alpha|\vec{x}|\alpha\rangle$$

$$\pi^{+}\vec{x}\pi = -\vec{x}$$

$$\vec{x}\pi = -\pi\vec{x}$$

$$\{\vec{x}, \pi\} = 0$$

Suppose $\pi|\vec{x}'\rangle = e^{i\delta}|-\vec{x}'\rangle$

$$\vec{x}\pi|\vec{x}'\rangle = -\pi\vec{x}|\vec{x}'\rangle$$

$$= -\pi(\vec{x}')|\vec{x}'\rangle$$

$$= (-\vec{x}')\pi|\vec{x}'\rangle$$

$e^{i\delta} = 1$ convention

$$\therefore \pi|\vec{x}'\rangle = e^{i\delta}|-\vec{x}'\rangle$$

unitary (arbitrary phase)

$\pi^2 = 1$ $\qquad \pi\pi|\vec{x}'\rangle = \pi|-\vec{x}'\rangle = |\vec{x}'\rangle$

$\therefore$ $\pi^2 = 1$

$\pi^{+}\pi = 1$ unitary

$\Rightarrow \pi^{+} = \pi \Rightarrow$ Hermitian.

Show that $\begin{cases} \{\pi, p\} = 0 \\ [\pi, \vec{L}] = 0, \quad \vec{L} = \vec{x} \times \vec{p} \end{cases}$

$$\Downarrow$$

parity & rotation commute

$$[R(\text{parity}), R(\text{rotation})] = 0.$$

$$[\pi, \mathcal{D}(R)] = 0.$$

$\Rightarrow$ generalization

$$[\pi, \vec{J}] = 0$$

$\vec{x}, \vec{p}$; polar vector. $\{\pi, \vec{x}\} = 0$

$\vec{L}, \vec{J}$; axial vector. $[\pi, \vec{L}] = 0$

$\vec{x} \cdot \vec{p}$; scalar $\{\pi, \vec{x} \cdot \vec{p}\} = 0$

$\vec{L} \cdot \vec{S}$; pseudoscalar $[\pi, \vec{L} \cdot \vec{S}] = 0$.

Wavefunctions under Parity

$\psi_\alpha(x) = \langle x|\alpha\rangle$

recall $\begin{cases} \pi^\dagger = \pi \\ \pi|x\rangle = |-x\rangle \end{cases} \rightarrow \quad \langle -x| = \langle x|\pi^\dagger$

$= \langle x|\pi$

$\therefore \quad \langle -x|\alpha\rangle$

$\psi_\alpha(-x) = \langle -x|\alpha\rangle = \langle x|\pi|\alpha\rangle$

$\pi^2 = 1$

if $|\alpha\rangle$ is an eigenket of π

then $\pi|\alpha\rangle = \lambda|\alpha\rangle$

$\pi^2|\alpha\rangle = \lambda^2|\alpha\rangle = |\alpha\rangle$

$\lambda^2 = 1 \rightarrow \lambda = \pm 1$

$\therefore \psi_\alpha(-x) = \langle x|\pi|\alpha\rangle = \pm\langle x|\alpha\rangle$

+ ; even parity

− ; odd parity

- $\{\pi, \vec{p}\} = 0$ ($\vec{p}$ is a polar vector)

  $\pi\vec{p} \neq \vec{p}\pi \Rightarrow$ no common eigenket.

  $\Rightarrow |\vec{p}\rangle$ is not an eigenket of π.

plane wave

$$\psi_\alpha(x) = e^{+\frac{i}{\hbar}\vec{p}\cdot\vec{x}} = \langle \vec{x}|\vec{p}\rangle.$$

$$\psi_\alpha(-x) = \langle -\vec{x}|\vec{p}\rangle = \langle \vec{x}|\pi|\vec{p}\rangle = \langle \vec{x}|-\vec{p}\rangle$$

$$= e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{x}}$$

- $[\pi, \vec{L}] = 0$ ($\vec{L}$ is an axial vector)

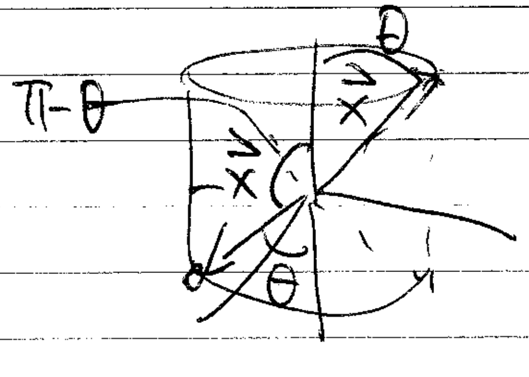
  $\pi\vec{L} = \vec{L}\pi \Rightarrow \exists$ common eigenket.

  $\Rightarrow |\cancel{\alpha}\, l, m\rangle$

$$R_n(r)\, Y_{lm}(\theta, \phi) = \langle \vec{x}|l, m\rangle$$

π $\quad \vec{r} \xrightarrow{\pi} -\vec{r}$

$$\begin{pmatrix} r \xrightarrow{\pi} r \\ \theta \xrightarrow{\pi} \pi-\theta \\ \phi \longrightarrow \phi+\pi \end{pmatrix}.$$



Theorem) If ~~suppose~~ $[H, \pi] = 0$ & nondegenerate

& $H|n\rangle = E_n|n\rangle$,

then $|n\rangle$ is ~~an~~ parity eigenket.

Note that $\pi^\dagger = \pi$; $\pi^2 = 1$.

$A = a + b\pi$ & a & b are complex #.

Suppose $[A, \pi] = 0$

$\pi A = a\pi + b$

$A\pi = a + b\pi = b\pi + a$.

$\pi A|n\rangle = \pi(a\pi + b)|n\rangle$

$= (a + b\pi)|n\rangle = a|n\rangle + b(\pi|n\rangle)$.

$\frac{1}{2}\left(f(x) + f(-x)\right) \Rightarrow$ even

$\frac{1}{2}\left(f(x) - f(-x)\right) \Rightarrow$ odd for any $f(x)$

$$\frac{1}{2}\left(\langle x|\alpha\rangle + \langle -x|\alpha\rangle\right) = \frac{1}{2}\left(\langle x|\alpha\rangle + \langle x|\pi|\alpha\rangle\right)$$

$$= \langle x| \frac{1}{2}\left(|\alpha\rangle + \pi|\alpha\rangle\right)$$

$$= \langle x| \boxed{\frac{1}{2}(1+\pi)} |\alpha\rangle$$

$$\frac{1}{2}\left(\langle x|\alpha\rangle - \langle -x|\alpha\rangle\right) = \langle x| \frac{1}{2}(1-\pi)|\alpha\rangle.$$

$|\alpha\rangle \to |n\rangle$

Therefore, $\frac{1}{2}(1+\pi)|\alpha\rangle$; even parity

$\frac{1}{2}(1-\pi)|\alpha\rangle$: odd parity.

$$H\,\tfrac{1}{2}(1\pm\pi)|\alpha\rangle = \tfrac{1}{2}(1\pm\pi)\,H|\alpha\rangle$$

$$= \tfrac{1}{2}(1\pm\pi)\,E_n|n\rangle$$

$$= E_n\left[\tfrac{1}{2}(1\pm\pi)|n\rangle\right]$$

$\therefore$ $\frac{1}{2}(1\pm\pi)|n\rangle$ and $|n\rangle$ ~~have~~ are degenerate. (the same energy E_n)

~~This contradicts to the assumption that.~~

~~Because~~ $|n\rangle$ is nondegenerate

Therefore, $|n\rangle$ and $\frac{1}{2}(1\pm\pi)|n\rangle$ are the same up to an overall factor.

$$\tfrac{1}{2}(1\pm\pi)|n\rangle \propto |n\rangle$$

$$\Rightarrow \pi|n\rangle \propto |n\rangle.$$

$$\left(\begin{aligned}\pi\,\tfrac{1}{2}(1\pm\pi) &= \tfrac{1}{2}(\pi\pm\pi^2) = \tfrac{1}{2}(\pi\pm1)\\ &= \pm\left[\tfrac{1}{2}(1\pm\pi)\right]\end{aligned}\right).$$

$$\therefore\ \boxed{\pi|n\rangle = \pm|n\rangle.}$$

(Example)

Simple harmonic oscillator

$|0\rangle$ ground state

$|1\rangle = a^\dagger|0\rangle$ first excited state

① show that $|0\rangle$ is even.

$$\langle x|0\rangle = \frac{1}{\sqrt{\pi\sqrt{x_0}}} e^{-\frac{1}{2}\left(\frac{x}{x_0}\right)^2},$$

$$x_0 = \sqrt{\frac{\hbar}{m\omega}}.$$

$$\langle -x|0\rangle = \langle x|\pi|0\rangle = \langle x|0\rangle.$$

② $|1\rangle = a^\dagger|0\rangle$. $\quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$

$|n\rangle \doteq \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$. $\quad a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right)$

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

$$\langle -x|1\rangle = \langle x|\pi|1\rangle \qquad \pi|0\rangle = |0\rangle$$

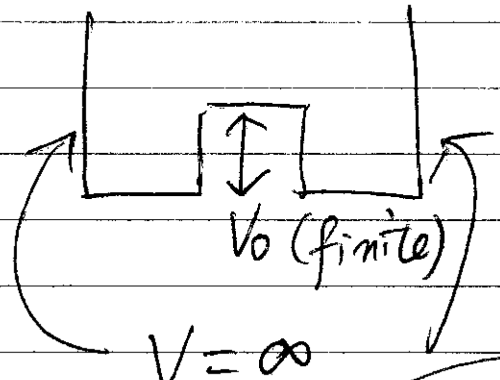
$$= \langle x|\pi\overbrace{\left(x - \frac{ip}{m\omega}\right)}^{a^\dagger}|0\rangle\sqrt{\frac{m\omega}{2\hbar}}$$

$$= \langle x|\pi\left(x - \frac{ip}{m\omega}\right)\pi^\dagger\pi|0\rangle\sqrt{\frac{m\omega}{2\hbar}}$$

$$-\langle x|1\rangle = \; = \langle x|\left[-\left(x - \frac{ip}{m\omega}\right)\right]\pi|0\rangle = -\langle x|a^\dagger|0\rangle$$

odd.

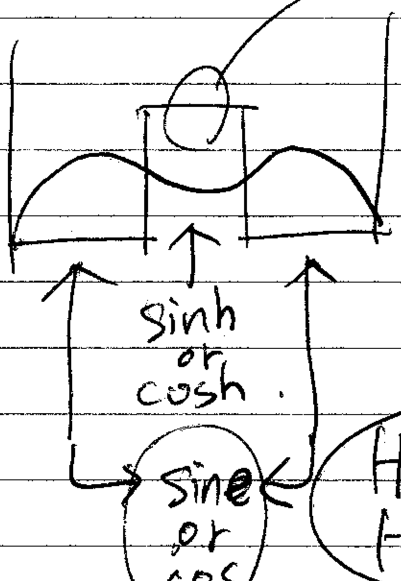
Symmetrical Double-Well potential

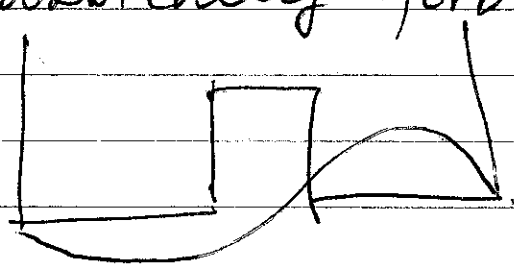


$V(x) = V(-x)$

$[H, \pi] = 0$.

classically forbidden.

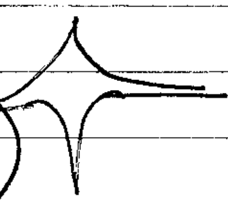




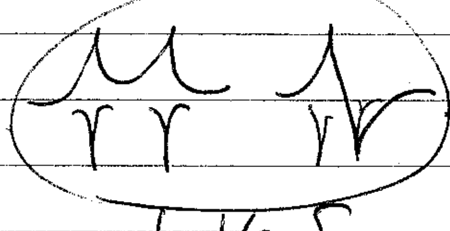
$E_S < E_A$.

$H|S\rangle = E_S|S\rangle$

$H|A\rangle = E_A|A\rangle$



, δ-function.



double δ.

Classically allowed.

$|S\rangle$, $\pi|S\rangle = |S\rangle$ Symmetric

$|A\rangle$, $\pi|A\rangle = -|A\rangle$ antisymmetric.

$|R\rangle \equiv \frac{1}{\sqrt{2}}(|S\rangle + |A\rangle)$ more probable on the right side

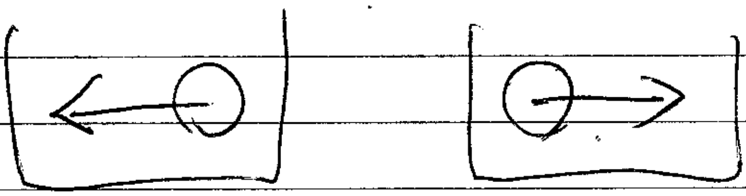
$|L\rangle = \frac{1}{\sqrt{2}}(|S\rangle - |A\rangle)$. left.

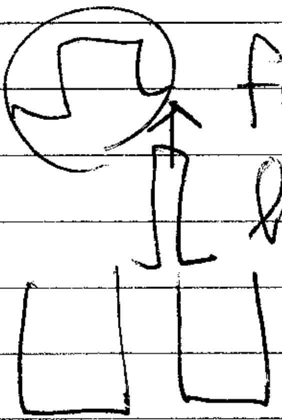
$|R\rangle$ & $|L\rangle$: not ~~an~~ eigenkets of π & H.

$$|\alpha(t=0)\rangle = |R\rangle$$
$$= \frac{1}{\sqrt{2}}(|S\rangle + |A\rangle)$$
$$|\alpha(t)\rangle = e^{-\frac{i}{\hbar}Ht}|\alpha(t=0)\rangle$$
$$= \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{\hbar}E_S t}|S\rangle + e^{-\frac{i}{\hbar}E_A t}|A\rangle\right)$$
$$= \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}E_S t}\left(|S\rangle + e^{-\frac{i}{\hbar}\underbrace{(E_A-E_S)}_{>0} t}|A\rangle\right).$$
$$\omega = \frac{E_A - E_S}{\hbar}$$
$$T = \frac{2\pi}{\omega}$$
at $t = T(n+\frac{1}{2})$ $\propto \frac{1}{\sqrt{2}}(|S\rangle - |A\rangle) = |L\rangle$

$\Rightarrow$ os $\quad |R\rangle \Longleftrightarrow |L\rangle$

oscillation!



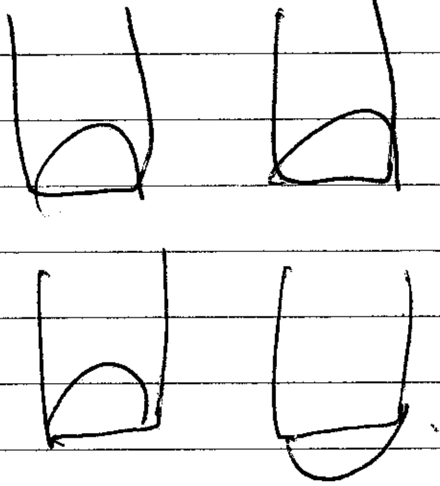


finite barrier $\Rightarrow$ tunneling.

$\ast$ as higher $\Rightarrow$ two isolated infinite potential wells

$\Rightarrow$ no tunneling

$\Rightarrow$ no oscillation.



sym $\Updownarrow$ antisym $\Rightarrow$ degenerate?

$\Downarrow$

$|S\rangle$

$|A\rangle$

energy eigenkets do not have to be eigenkets of π. because $|s\rangle$ & $|A\rangle$ are degenerate.

Parity - selection rule..

$$\pi|\alpha\rangle = \varepsilon_\alpha|\alpha\rangle$$

$$\pi|\beta\rangle = \varepsilon_\beta|\beta\rangle. \qquad \varepsilon_\alpha, \varepsilon_\beta = \pm 1.$$

$$\langle\beta|\vec{x}|\alpha\rangle = \langle\beta|\pi\vec{x}\pi^{\dagger}|\alpha\rangle = -\cancel{\langle\beta|\vec{x}|\alpha\rangle}$$

$$= (\pi^{\dagger}|\beta\rangle)^{\dagger}\vec{x}$$

$$\begin{aligned}\langle\beta|\vec{x}|\alpha\rangle &= \langle\beta|\pi^{\dagger}\vec{x}\,\pi^{2}|\alpha\rangle \\ &= (\pi|\beta\rangle)^{\dagger}(\pi\vec{x}\pi^{\dagger})(\pi|\alpha\rangle) \\ &= \langle\beta|\pi^{\dagger}(-\vec{x})\pi|\alpha\rangle = -\varepsilon_\alpha\varepsilon_\beta\langle\beta|\vec{x}|\alpha\rangle.\end{aligned}$$

$(1+\varepsilon_\alpha \varepsilon_\beta)\langle\beta|\vec{x}|\alpha\rangle = 0$

if $1+\varepsilon_\alpha\varepsilon_\beta \neq 0$, then $\langle\beta|\vec{x}|\alpha\rangle = 0$.

$(\varepsilon_\alpha\varepsilon_\beta \neq -1)$

~~Wigner showed that~~

If $[H,\pi]=0$ and $|n\rangle$ are nondegenerate,
then $\langle n|\vec{x}|n\rangle = 0$.

no permanent electric dipole moment

Parity Non-conservation

$\langle\vec{s}\rangle\cdot\vec{p} \Rightarrow$ pseudo scalar

electromagnetic interaction $\Rightarrow [H,\pi]=0$

However, weak interaction
does not conserve parity
(T.D. Lee & C.N. Yang).

4.4 Time Reversal. $t \to -t$.

$$m \frac{d^2 \vec{x}}{dt^2} = -\vec{\nabla} V$$

$t \to -t$

$$\left.\begin{aligned} m \frac{d^2 x}{d(-t)^2} &= -\vec{\nabla} V \\ &= m \frac{d^2 x}{dt^2} \end{aligned}\right\} \text{ time-reversal invariant.}$$

If $x(t)$ is a solution, then $x(-t)$ is also a solution?

$$m\ddot{x} + b\dot{x} + kx = 0.$$

($t \to -t$: $m\ddot{x} - b\dot{x} + kx = 0.$
not invariant!)

Maxwell's equations

$$\begin{cases}\vec{\nabla}\cdot\vec{E}=\rho\\ \vec{\nabla}\times\vec{B}-\frac{1}{c}\frac{\partial\vec{E}}{\partial t}=\frac{1}{c}\vec{J}\\ \vec{\nabla}\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\\ \vec{\nabla}\cdot\vec{B}=0.\end{cases}\quad\xrightarrow{t\to -t}\quad\begin{array}{l}\vec{\nabla}\cdot\vec{E}=\rho\\ -\vec{\nabla}\times\vec{B}+\frac{1}{c}\frac{\partial\vec{E}}{\partial t}=-\frac{1}{c}\vec{J}\\ \vec{\nabla}\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\\ \vec{\nabla}\cdot\vec{B}=0.\end{array}$$

$e=|e|$

$q_e=-e$

Lorentz force: $\vec{F}=e\left(\vec{E}+\frac{\vec{v}}{c}\times\vec{B}\right).$

$$\vec{E}=-\vec{\nabla}\phi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

$$\vec{B}=\vec{\nabla}\times\vec{A}$$

$(\vec{A}\propto\vec{J})$

$\vec{F}=-e\left(\vec{E}+\frac{\vec{v}}{c}\times\vec{B}\right)$

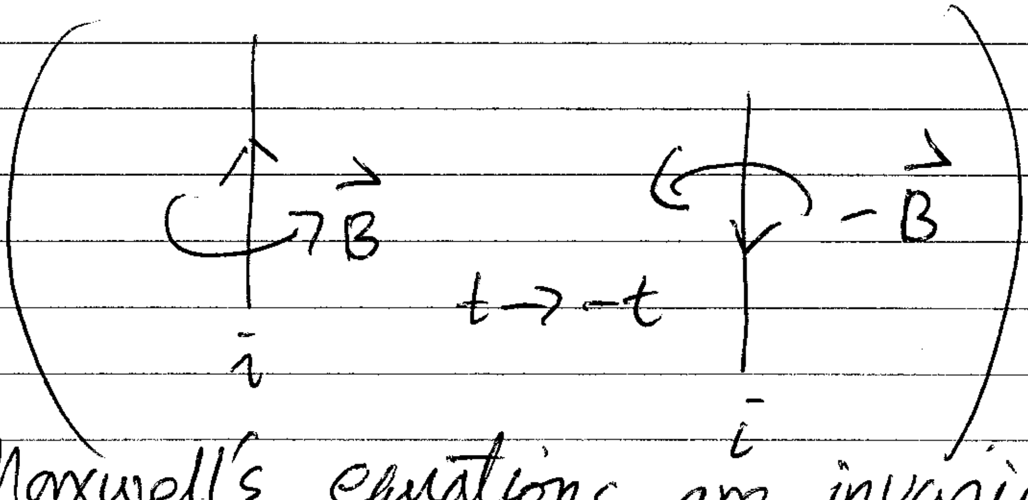
invariant

$$\begin{Bmatrix}\phi\xrightarrow{T}\phi\\ A\xrightarrow{T}-\vec{A}\end{Bmatrix}$$

$\vec{E}\to\vec{E}$ $\quad\rho\to\rho$

$\vec{B}\to-\vec{B}$ $\quad\vec{J}\to-\vec{J}$

$\vec{v}\to-\vec{v}$



Maxwell's equations are invariant under time reversal.

Schrödinger equation

$$i\hbar \frac{\partial \psi(t,x)}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi$$

$$-i\hbar \frac{\partial \psi(-t,x)}{\partial t} = \left(\quad '' \quad \right)\psi(-t,x)$$

But

$$i\hbar \frac{\partial}{\partial t}\psi^*(-t,x) = \left(\quad '' \quad \right)\psi^*(-t,x).$$

if $\psi(t,\vec{x})$ is a solution,

then $\psi^*(-t,\vec{x})$ is another solution.

$$e^{i(kx-\omega t)} \longrightarrow \left[e^{i(kx+\omega t)}\right]^*$$

$$e^{-i(-kx-\omega t)}$$

"complex conjugate" is related to the time-reversed wavefunction

$$i\hbar \frac{\partial \psi(t,x)}{\partial t} = \left(\frac{-\hbar^2}{2m}\nabla^2 + V(x)\right)\psi(t,x)$$

$$-i\hbar \frac{\partial \psi(x,-t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right]\psi(-t,x)$$

complex conjugate

$$i\hbar \frac{\partial}{\partial t}\psi^*(-t,x) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right]\psi^*(-t,x).$$

$$\psi(t,x) = u_n(x)\, e^{-\frac{i}{\hbar}E_n t}$$

$$\psi^*(-t,x) = u_n^*(x)\, e^{-\frac{i}{\hbar}E_n t}$$

time reversal has something to do with the complex conjugation.

at $t=0$

$$\psi(t=0,x) = u_n(x) = \langle x|\alpha\rangle$$

$$\psi^*(t=0,x) = u_n^*(x) = \langle \alpha|x\rangle = \langle x|\alpha\rangle^*$$