Quantum Mechanics 1.

Assignment 2.

T.A solution.

1. Consider a wave function of the form

$$\psi(x) = A\exp(-ax^2),$$

where A and a are positive constants.

(a) Express A in terms of a.

⇒ i) wave function $\psi(x)$ 가 normalize 되어야 한다는 조건을 이용하면, A 와 a의 관계를 구할 수 있다.

$$\int_{-\infty}^{\infty} dx\, |\psi|^2 = 1.$$

A 와 a 모두 real positive.

$$\int_{-\infty}^{\infty} dx\, A^2 \exp(-2ax^2) = A^2 \cdot \int_{-\infty}^{\infty} dx\, \exp(-2ax^2)$$

$$\boxed{\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}} \quad = A^2 \sqrt{\frac{\pi}{2a}} = 1$$

$$\therefore A^2 = \left(\frac{2a}{\pi}\right)^{1/2} \longrightarrow A = \left(\frac{2a}{\pi}\right)^{1/4}$$

(b) Compute the wave function $\phi(p)$ in momentum space corresponding to $\psi(x)$.

⇒ i) $$\phi(p) = \int dx \cdot \frac{e^{-i\frac{p}{\hbar}x}}{\sqrt{2\pi\hbar}} \cdot \psi(x)$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int dx\, e^{-i\frac{p}{\hbar}x} A e^{-ax^2}$$

$$= \frac{A}{\sqrt{2\pi\hbar}} \int dx\, e^{-a\left(x^2 + \frac{ip}{a\hbar}x\right)}$$

ii) $\phi(p) = \frac{A}{\sqrt{2\pi\hbar}} \int dx \, e^{-a\left(x+\frac{ip}{2a\hbar}\right)^2 - a\frac{p^2}{4a^2\hbar^2}}$

$$= \frac{A}{\sqrt{2\pi\hbar}} e^{-\frac{p^2}{4a\hbar^2}} \int dx \, e^{-a\left(x+\frac{ip}{2a\hbar}\right)^2}$$

이때.. $f(z) = e^{-az}$ 는 complex plane 에서

pole 이 없으므로..

$$\int dx \, e^{-a\left(x+\frac{ip}{2a\hbar}\right)^2} = \int dx \, e^{-ax^2} = \sqrt{\frac{\pi}{a}} \text{ 이다.}$$

$$\therefore \phi(p) = \frac{A}{\sqrt{2\pi\hbar}} \cdot e^{-\frac{p^2}{4a\hbar^2}} \cdot \sqrt{\frac{\pi}{a}}$$

$$= \frac{A}{\sqrt{2a\hbar}} \cdot e^{-\frac{p^2}{4a\hbar^2}}$$

(C) Compute $\langle x^2 \rangle$ and $\langle x \rangle^2$ using $\psi(x)$ and $\phi(p)$.

$\Rightarrow$ i) 먼저 configuration space 에서.. $\langle x^2 \rangle$, $\langle x \rangle^2$ 구해보자.

$$\langle x \rangle = \int_{-\infty}^{\infty} dx \; \psi^*(x) \cdot x \cdot \psi(x)$$

$$= \int_{-\infty}^{\infty} dx \cdot x \cdot |\psi|^2 = A^2 \cdot \int_{-\infty}^{\infty} dx \cdot \underbrace{x}_{\text{odd parity}} \cdot \underbrace{\exp(-2ax^2)}_{\text{even parity}}$$

$= 0.$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx \cdot \psi^*(x) \cdot x^2 \, \psi(x)$$

$$= \int_{-\infty}^{\infty} dx \cdot x^2 \cdot |\psi|^2$$

$$= \int_{-\infty}^{\infty} dx \cdot x^2 \cdot A^2 \exp(-2ax^2)$$

$$= A^2 \cdot \int_{-\infty}^{\infty} dx \cdot x^2 \exp(-2ax^2)$$

$$= A^2 \cdot \int_{-\infty}^{\infty} dx \cdot \left(-\frac{1}{2}\frac{\partial}{\partial a}\right) \exp(-2ax^2)$$

$$= -\frac{A^2}{2} \cdot \frac{d}{da} \cdot \underbrace{\int_{-\infty}^{\infty} dx \cdot \exp(-2ax^2)}$$

$$\hookrightarrow \left(\frac{\pi}{2a}\right)^{1/2} = \left(\frac{\pi}{2}\right)^{1/2} \cdot a^{-1/2}$$

$$= -\frac{A^2}{2} \cdot \left(\frac{\pi}{2}\right)^{1/2} \cdot \left(-\frac{1}{2}\right) \cdot a^{-3/2}$$

$$= \frac{A^2}{4}\left(\frac{\pi}{2}\right)^{1/2} \cdot a^{-3/2} = \frac{1}{4} \cdot \left(\frac{2a}{\pi}\right)^{1/2} \cdot \left(\frac{\pi}{2}\right)^{1/2} \left(\frac{1}{a^3}\right)^{1/2}$$

$$= \frac{1}{4a}$$

$$\therefore \langle x^2 \rangle = \frac{1}{4a} \qquad \langle x \rangle^2 = 0 .$$

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ii) 다음으로.. momentum space 에서 $\langle x^2 \rangle$, $\langle x \rangle^2$ 구해보자.

$$\langle x \rangle = \int_{-\infty}^{\infty} dp\ \phi^*(p) \left(-\frac{\hbar}{i}\frac{d}{dp}\right) \phi(p)$$

$$= \frac{A^2}{2a\hbar}\left(-\frac{\hbar}{i}\right) \cdot \int_{-\infty}^{\infty} dp \cdot e^{-\frac{p^2}{4a\hbar^2}} \cdot \left(-\frac{p}{2a\hbar^2}\right) e^{-\frac{p^2}{4a\hbar^2}}$$

$$= +\left(\frac{A}{2a\hbar}\right)^2 \cdot \frac{1}{i} \cdot \int_{-\infty}^{\infty} dp\ \underbrace{p}_{\text{odd}}\ \underbrace{e^{-\frac{p^2}{2a\hbar^2}}}_{\text{even}} = 0.$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dp\ \phi^*(p) \left(-\frac{\hbar}{i}\frac{d}{dp}\right)^2 \phi(p).$$

$$= (-\hbar^2) \cdot \int_{-\infty}^{\infty} dp\ \phi^*(p) \left(\frac{d}{dp}\right)^2 \phi(p)$$

$$\boxed{\begin{aligned} &\frac{d\phi(p)}{dp} = \frac{A}{\sqrt{2a\hbar}} \cdot \left(-\frac{p}{2a\hbar^2}\right) e^{-\frac{p^2}{4a\hbar^2}} \\ &\frac{d^2\phi(p)}{dp^2} = \frac{A}{\sqrt{2a\hbar}} \left(-\frac{p}{2a\hbar^2}\right)^2 e^{-\frac{p^2}{4a\hbar^2}} \\ &\qquad + \frac{A}{\sqrt{2a\hbar}} \cdot \left(-\frac{1}{2a\hbar^2}\right) e^{-\frac{p^2}{4a\hbar^2}} \end{aligned}}$$

$$= (-\hbar^2) \int_{-\infty}^{\infty} dp \left[\frac{A^2}{2a\hbar}\left(\frac{1}{(2a\hbar^2)^2} p^2 e^{-\frac{p^2}{2a\hbar^2}} - \frac{1}{2a\hbar^2} \cdot e^{-\frac{p^2}{2a\hbar^2}}\right)\right]$$

$$= \frac{A^2}{2a\hbar}\cdot(-\hbar^2)\left\{\frac{1}{(2a\hbar^2)^2}\cdot\int_{-\infty}^{\infty} dp\ p^2 e^{-\frac{p^2}{2a\hbar^2}}\right.$$

$$\left. - \frac{1}{2a\hbar^2}\cdot\int_{-\infty}^{\infty} dp\ e^{-\frac{p^2}{2a\hbar^2}}\right\}$$

$$\boxed{\begin{aligned}
&\textcircled{*}\ \cdot \int_{-\infty}^{\infty} dp\ e^{-\frac{p^2}{2a\hbar^2}} = \sqrt{\frac{\pi}{1/2a\hbar^2}} = \hbar\sqrt{2\pi a} \\
&\cdot \int_{-\infty}^{\infty} dp\ p^2 e^{-\frac{p^2}{2a\hbar^2}} = \int_{-\infty}^{\infty} dp\left(-\frac{\partial}{\partial\alpha}\right)e^{-\alpha p^2} \\
&\qquad\qquad \text{where } \alpha = \frac{1}{2a\hbar^2} \\
&= -\frac{d}{d\alpha}\int_{-\infty}^{\infty} dp\ e^{-\alpha p^2} \\
&= -\frac{d}{d\alpha}\cdot\sqrt{\frac{\pi}{\alpha}} = -\sqrt{\pi}\cdot\frac{d}{d\alpha}(\alpha^{-1/2}) \\
&= \frac{1}{2}\sqrt{\pi}\ \alpha^{-3/2} = \frac{1}{2}\sqrt{\pi}\cdot\left(\frac{1}{2a\hbar^2}\right)^{-3/2} \\
&= \frac{1}{2}\sqrt{\pi}\ (2a\hbar^2)^{3/2} = \frac{1}{2}(2a\hbar^2)\cdot(2\pi a\hbar^2)^{1/2}
\end{aligned}}$$

$$= -\frac{A^2\hbar}{2a}\left\{\frac{1}{(2a\hbar^2)^2}\cdot\frac{1}{2}\cdot(2a\hbar^2)(2\pi a)^{1/2}\hbar\right.$$

$$\left. - \frac{1}{2a\hbar^2}\cdot(2\pi a)^{1/2}\cdot\hbar\right\}$$

$$= \frac{A^2\cancel{\hbar}}{2a}\cdot\frac{1}{2}\cdot\frac{1}{2a\cancel{\hbar^2}}\cdot(2\pi a)^{1/2}\cancel{\hbar} = \frac{A^2}{8a^2}\cdot(2\pi a)^{1/2}$$

$$\therefore \langle x^2 \rangle = \frac{1}{8a^2} \cdot (2\pi a)^{1/2} \cdot \left(\frac{2a}{\pi}\right)^{1/2}$$

$$= \frac{1}{8a^2} \cdot 2a = \frac{1}{4a}$$

$$\therefore \langle x^2 \rangle = \frac{1}{4a}, \quad \langle x \rangle^2 = 0$$

↳ configuration space 결과 동일.

(d) Compute $\langle p^2 \rangle$ and $\langle p \rangle^2$ using $\psi(x)$ and $\phi(p)$.

⇒ i) 먼저 configuration space에서 계산.

$$\langle p \rangle = \int dx\, \psi^*(x) \left(\frac{\hbar}{i}\frac{d}{dx}\right) \psi(x)$$

$$= A^2 \frac{\hbar}{i} \cdot \int dx\, \exp(-ax^2) \cdot (-2ax) \exp(-ax^2)$$

$$= A^2 \frac{\hbar}{i} (-2a) \int dx\, \underbrace{x}_{\text{odd}} \cdot \underbrace{\exp(-2ax^2)}_{\text{even}} = 0.$$

$$\langle p^2 \rangle = \int dx\, \psi^*(x) \left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 \psi(x)$$

$$= A^2 (-\hbar^2) \int dx\, \exp(-ax^2) \left[(-2ax)^2 - 2a\right] \exp(-ax^2)$$

$$= -A^2\hbar^2 \left\{ (+4a^2) \int dx\, x^2 \exp(-2ax) \right.$$

$$\left. -2a \int dx\, \exp(-2ax) \right\}$$

$$= -A^2\hbar^2 \left[4a^2 \cdot \frac{1}{4a}\sqrt{\frac{\pi}{2a}} - 2a \cdot \sqrt{\frac{\pi}{2a}} \right]$$

$$= A^2\hbar^2 \cdot a\sqrt{\frac{\pi}{2a}} = \left(\frac{2a}{\pi}\right)^{1/2} \hbar^2 \cdot a\left(\frac{\pi}{2a}\right)^{1/2}$$

$$= \hbar^2 a$$

$$\therefore \langle p^2 \rangle = \hbar^2 a, \quad \langle p \rangle = 0.$$

(ii) 다음으로.. momentum space 에서 계산.

$$\langle p \rangle = \int dp\, \phi^*(p)\, p\, \phi(p) = \int dp\, \underbrace{p}_{\text{odd}} \cdot \underbrace{|\phi(p)|^2}_{\text{even}} = 0$$

$$\langle p^2 \rangle = \int dp\, \phi^*(p)\, p^2\, \phi(p)$$

$$= \frac{A^2}{2a\hbar} \int dp\, p^2 \cdot e^{-\frac{p^2}{2a\hbar^2}}$$

$$= \frac{A^2}{2a\hbar}\, \frac{1}{2} \cdot (2a\hbar^2)\,(2\pi a)^{1/2}\,\hbar.$$

$$= \frac{\hbar^2}{2} \left(\frac{2a}{\pi}\right)^{1/2} (2\pi a)^{1/2}. = \hbar^2 a$$

$\therefore \langle p^2 \rangle = \hbar^2 a. \qquad \langle p \rangle = 0.$

$\therefore$ configuration space 에서 구한 결과와 일치!

(e) Verify the uncertainty relation $\Delta x \cdot \Delta p \geq \hbar/2$.

$\Rightarrow$ i) $\langle x^2 \rangle = \dfrac{1}{4a} \qquad \langle x \rangle^2 = 0$

$$\therefore \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{1}{2\sqrt{a}}$$

ii) $\langle p^2 \rangle = \hbar^2 a \qquad \langle p \rangle^2 = 0.$

$$\therefore \Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \hbar\sqrt{a}.$$

iii)

$$\therefore \Delta x \cdot \Delta p = \frac{1}{2\sqrt{a}} \cdot \hbar\sqrt{a} = \frac{\hbar}{2} \geq \frac{\hbar}{2}\,!$$

2. Find the Fourier transforms of the following functions:

(a) $f(x) = \begin{cases} x, & |x|<1 \\ 0, & |x| \geq 1 \end{cases}$

$\Rightarrow$ i) $\hat{f}(k) = \int dx\, e^{-ikx} f(x)$

$= \cancel{\int_{-\infty}^{-1} dx\, e^{-ikx} \cdot 0}$

$+ \int_{-1}^{1} dx\, e^{-ikx} \cdot x + \cancel{\int_{1}^{\infty} dx\, e^{-ikx} \cdot 0}$

$= \int_{-1}^{1} dx\, \underset{u'}{e^{-ikx}} \cdot \underset{v}{x}$) integration by parts.

$= -\frac{e^{-ikx}}{ik} \cdot x \Big]_{-1}^{1}$

$- \int_{-1}^{1} dx \left(-\frac{e^{-ikx}}{ik}\right)$

$= -\frac{e^{-ik}}{ik} - \frac{e^{ik}}{ik} + \frac{1}{ik} \int_{-1}^{1} dx\, e^{-ikx}$

$= -\frac{2}{ik} \cdot \frac{e^{ik}+e^{-ik}}{2} + \frac{1}{ik \cdot (-ik)} e^{-ikx} \Big]_{-1}^{1}$

$= \frac{2i}{k} \cos(k) + \frac{1}{k^2} \cdot (e^{-ik} - e^{ik})$

$$= \frac{2i}{k}\cos(k) - \frac{2i}{k^2}\cdot \sin(k)$$

$$= 2i \cdot \frac{k\cos(k) - \sin(k)}{k^2}$$

ii) $\therefore\ \hat{f}(k) = 2i \cdot \frac{k\cos(k) - \sin(k)}{k^2}.$

(b) $f(x) = \begin{cases} |x|, & |x| < 1. \\ 0, & |x| \geq 1. \end{cases}$

$\Rightarrow$ i) $\hat{f}(k) = \int dx\, e^{-ikx} f(x)$

$$= \int_{-1}^{1} dx\, e^{-ikx} \cdot |x|$$

$$= \int_{-1}^{0} dx\, e^{-ikx} \cdot (-x) + \int_{0}^{1} dx\, e^{-ikx} \cdot x$$

$$= -\int_{-1}^{0} dx\, e^{-ikx} \cdot x + \int_{0}^{1} dx\, e^{-ikx} \cdot x$$

$$= -\frac{d}{d(-ik)} \int_{-1}^{0} dx\, e^{-ikx} + \frac{d}{d(-ik)} \int_{0}^{1} dx\, e^{-ikx}$$

$$= -i \frac{d}{dk}\left(\frac{1-e^{ik}}{(-ik)}\right) + i \cdot \frac{d}{dk}\left(\frac{e^{-ik}-1}{(-ik)}\right)$$

$$= \frac{d}{dk}\left(\frac{1-e^{ik}}{k}\right) - \frac{d}{dk}\left(\frac{e^{-ik}-1}{k}\right)$$

$$= \frac{d}{dk}\left(\frac{2-e^{ik}-e^{-ik}}{k}\right)$$

$$= 2 \cdot \frac{d}{dk}\left(\frac{1-\cos k}{k}\right) = 2 \cdot \left[-\frac{1}{k^2} \cdot (1-\cos k) + \frac{1}{k}(+\sin k)\right]$$

$$= 2 \cdot \frac{\cos k - 1 + k \sin k}{k^2}$$

$$\therefore \hat{f}(k) = 2 \cdot \frac{\cos k - 1 + k \sin k}{k^2}$$

(c) $f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x| \geq 1 \end{cases}$

→ i) $\hat{f}(k) = \int_{-1}^{1} dx\, e^{-ikx} \cdot (1-|x|)$

$$= \int_{-1}^{1} dx\, e^{-ikx} - \underbrace{\int_{-1}^{1} dx\, e^{-ikx} |x|}_{\text{(b)에서 구함.}}$$

$$= \frac{1}{-ik} \cdot (e^{-ik} - e^{ik}) - 2 \cdot \frac{\cos k - 1 + k \sin k}{k^2}$$

$$= \frac{2}{k} \sin k - 2 \cdot \frac{\cos k - 1 + k \sin k}{k^2}$$

$$= 2 \cdot \frac{1-\cos k}{k^2}$$

$\therefore \hat{f}(k) = 2 \cdot \frac{1-\cos k}{k^2}$

3. Consider the wave packet at $t=0$,

$$\psi(x,0) = A \exp\left[-(|x|/L) + i p_0 x/\hbar\right].$$

(a) Normalize $\psi(x,0)$.

$\Rightarrow$ i) $\int_{-\infty}^{\infty} dx\, |\psi(x,0)|^2$ (p_0 : real)

$$= |A|^2 \cdot \int_{-\infty}^{\infty} dx\, \underbrace{\exp\left[-2|x|/L\right]}_{\text{even}}$$

$$= |A|^2 \cdot 2 \int_0^{\infty} \exp(-2x/L)$$

$$= 2|A|^2 \cdot \left(-\frac{L}{2}\right) \cdot \exp(-2x/L)\Big]_0^{\infty}$$

$$= -|A|^2 \cdot L \cdot (0-1) = |A|^2 \cdot L = 1.$$

$\therefore$ $A = \dfrac{1}{\sqrt{L}}$

(b) Calculate $\phi(p,0)$ and $\phi(p,t)$. Verify that each is normalized.

$\Rightarrow$ i)

$$\phi(p,0) = \int dx \, \frac{1}{\sqrt{2\pi\hbar}} \cdot e^{-i\frac{p}{\hbar}x} \cdot \psi(x,0)$$

$$= \frac{A}{\sqrt{2\pi\hbar}} \cdot \int dx \, e^{-i\frac{p}{\hbar}x} \cdot \exp\left[-(|x|/L) + ip_0 x/\hbar\right]$$

$$= \frac{A}{\sqrt{2\pi\hbar}} \int dx \, \exp\left[-(|x|/L) + i(p_0-p)x/\hbar\right]$$

$$= \frac{A}{\sqrt{2\pi\hbar}} \left[\int_0^\infty dx \, \exp\left[-x/L + i(p_0-p)x/\hbar\right] \right.$$

change of variable

$$\left. + \int_{-\infty}^0 dx \, \exp\left[x/L + i(p_0-p)x/\hbar\right] \right\}$$

$$= \frac{A}{\sqrt{2\pi\hbar}} \left\{ \int_0^\infty dx \, \exp\left(\left[-\frac{1}{L} + i\frac{(p_0-p)}{\hbar}\right]x\right) \right.$$

$$\left. + \int_0^\infty dx \, \exp\left(\left[-\frac{1}{L} - i\frac{p_0-p}{\hbar}\right]x\right) \right\}$$

$$= \frac{A}{\sqrt{2\pi\hbar}} \int_0^\infty dx \left\{ \exp\left(\left[-\frac{1}{2} + i\frac{(p_0-p)}{\hbar}\right]x\right) \right.$$

$$\left. + \exp\left(\left[-\frac{1}{2} - i\frac{p_0-p}{\hbar}\right]x\right) \right\}.$$

ii) 이때.. exponential 함수는.. analytic 함수이므로..

$$\phi(P,0) = \frac{A}{\sqrt{2\pi\hbar}}\left[\frac{0-1}{-\frac{1}{L}+i\frac{(P_0-P)}{\hbar}} + \frac{0-1}{-\frac{1}{L}-i\frac{P_0-P}{\hbar}}\right]$$

$$= \frac{A}{\sqrt{2\pi\hbar}}\left[\frac{1}{\frac{1}{L}-i\frac{(P_0-P)}{\hbar}} + \frac{1}{\frac{1}{L}+i\frac{(P_0-P)}{\hbar}}\right]$$

$$= \frac{A}{\sqrt{2\pi\hbar}} \cdot \frac{2\cdot\frac{1}{L}}{\left(\frac{1}{L}\right)^2+\left(\frac{P_0-P}{\hbar}\right)^2}$$

$$= \sqrt{\frac{2}{\pi\hbar L}} \cdot \frac{\left(\frac{1}{L}\right)}{\left(\frac{1}{L}\right)^2+\left(\frac{P_0-P}{\hbar}\right)^2}.$$

$$\therefore \phi(P,0) = \sqrt{\frac{2}{\pi\hbar L}} \; \frac{\left(\frac{1}{L}\right)}{\left(\frac{1}{L}\right)^2+\left(\frac{P_0-P}{\hbar}\right)^2}$$

iii) $\phi(p,0)$ 가 normalized 인지 확인!

$$\int_{-\infty}^{\infty} dp\ \phi^*(p,0)\ \phi(p,0)$$

$$= \left(\frac{2}{\pi\hbar L}\right) \cdot \int_{-\infty}^{\infty} dp \cdot \left[\frac{(1/L)}{(1/L)^2 + (p_0 - p/\hbar)^2}\right]^2$$

$$= \left(\frac{2}{\pi\hbar L}\right) \cdot \int_{-\infty}^{\infty} dp\ \frac{(1/L)^2}{(1/L)^4} \cdot \left[\frac{1}{1 + \left(\frac{(p_0-p)L}{\hbar}\right)^2}\right]^2$$

$$= \frac{2L}{\pi\hbar} \int_{-\infty}^{\infty} dp \left[\frac{1}{1 + \left(\frac{(p_0-p)L}{\hbar}\right)^2}\right]^2$$

Setting $k = \frac{(p_0-p)L}{\hbar}$, $dk = -\frac{L}{\hbar}dp$,

$$= \frac{2L}{\pi\hbar} \int_{-\infty}^{\infty} dk \cdot \frac{\hbar}{L} \left(\frac{1}{1+k^2}\right)^2$$

$$= \frac{2}{\pi} \cdot \int_{-\infty}^{\infty} dk \left(\frac{1}{1+k^2}\right)^2$$

Setting $k = \tan\theta$, $dk = \sec^2\theta\, d\theta$

$$= \frac{2}{\pi} \cdot \int_{-\pi/2}^{\pi/2} \sec^2\theta\, d\theta \cdot \frac{1}{(1+\tan^2\theta)^2}$$

$$= \frac{2}{\pi} \cdot \int_{-\pi/2}^{\pi/2} d\theta \cdot \sec^2\theta \cdot \frac{1}{\sec^4\theta}$$

$$= \frac{2}{\pi} \cdot \int_{-\pi/2}^{\pi/2} d\theta \ \cos^2\theta$$

$$\hookrightarrow = \frac{1-\cos 2\theta}{2}$$

$$= \frac{2}{\pi} \cdot \frac{\pi}{2} = 1 .$$

$\therefore$ Normalized!

iv) 이제 $\phi(p,t)$ 를 찾아보자.

wavefunction의 time evolution을 알기위해서는..
hamiltonian을 알아야 한다. 여기서는

$H=\frac{p^2}{2m}$, 즉.. free particle state라 하자.

$$\therefore \psi(x,t) = e^{-i\frac{H}{\hbar}t}\,\psi(x,0)$$

$$= e^{-i\frac{H}{\hbar}t}\int dp\,\frac{1}{\sqrt{2\pi\hbar}}\cdot e^{i\frac{p}{\hbar}x}\,\phi(p,0)$$

$$= \int dp\cdot\frac{1}{\sqrt{2\pi\hbar}}\cdot e^{-i\frac{H}{\hbar}t}\,e^{i\frac{p}{\hbar}x}\,\phi(p,0)$$

$$= \int dp\cdot\frac{1}{\sqrt{2\pi\hbar}}\,e^{-i\frac{p^2}{2m\hbar}t}\,e^{i\frac{p}{\hbar}x}\,\phi(p,0)$$

$$= \int dp\cdot\frac{1}{\sqrt{2\pi\hbar}}\cdot e^{i\frac{p}{\hbar}x}\,\underbrace{e^{-i\frac{p^2}{2m\hbar}t}\,\phi(p,0)}_{\phi(p,t)\,..\,\ddot{}}$$

$$\boxed{\phi(p,t) = e^{-i\frac{p^2}{2m\hbar}t}\cdot\sqrt{\frac{2}{\pi\hbar L}}\;\frac{\left(\frac{1}{L}\right)}{\left(\frac{1}{L}\right)^2+\left(\frac{p_0-p}{\hbar}\right)^2}}$$

($\phi(p,0)$ 와 phase 차이만 있으므로.. normalization 동일!)

(c) Examine the width of the wave packets in configuration space and momentum space at $t=0$ and verify that the uncertainty relations are satisfied.

⇒ i) . configuration space wavefunction

$$\psi(x,0) = \frac{1}{\sqrt{L}} \exp\left[-|x|/L + ip_0 x/\hbar\right].$$

• momentum space wavefunction

$$\phi(p,0) = \sqrt{\frac{2}{\pi\hbar L}} \frac{(1/L)}{(1/L)^2 + (p_0 - p/\hbar)^2}$$

ii) . the width of the wave packet in configuration space

$$= \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

• the width of the wave packet in momentum space.

$$= \Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

그럼 이제 $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$, $\langle p^2 \rangle$ 을 찾아보자 !

iii) $\langle x \rangle = \int dx\ \psi^*(x)\ x\ \psi(x)$

$$= \int dx\ \underbrace{x}_{\text{odd}} \cdot \frac{1}{L} \underbrace{\exp[-2|x|/L]}_{\text{even}} = 0.$$

$$\langle x^2 \rangle = \int dx\ \psi^*(x) \cdot x^2\ \psi(x)$$

$$= \int dx \cdot x^2 \cdot \frac{1}{L} \exp[-2|x|/L]$$

$$= \frac{1}{L} \cdot \int_{-\infty}^{\infty} dx\ \underbrace{x^2 \exp[-2|x|/L]}_{\text{even}}$$

$$= \frac{2}{L} \cdot \int_0^{\infty} dx \cdot x^2 \exp[-2x/L] = \frac{2}{L} \cdot 2 \cdot \frac{1}{(2/L)^3}$$

$$= \frac{4}{L} \cdot \frac{L^3}{8} = \frac{L^2}{2}.$$

$$\int_0^{\infty} dx\ x^2 \exp(-\alpha x)$$

$$= \left(-\frac{d}{d\alpha}\right)^2 \int_0^{\infty} dx\ \exp(-\alpha x)$$

$$= \left(\frac{d}{d\alpha}\right)^2 \cdot \left. -\frac{1}{\alpha} \exp(-\alpha x) \right]_0^{\infty}$$

$$= \left(\frac{d}{d\alpha}\right)^2 \cdot \frac{1}{\alpha} = \frac{d}{d\alpha}\left(-\frac{1}{\alpha^2}\right) = 2\frac{1}{\alpha^3}$$

iv) $\langle p \rangle = \int dp\ \phi^*(p)\, p\, \phi(p)$

$$= \frac{2}{\pi\hbar L} \cdot \int_{-\infty}^{\infty} dp \left[\frac{(1/L)}{(1/L)^2 + (P_0 - P/\hbar)^2} \right]^2 p$$

$$= \frac{2}{\pi\hbar L} \cdot \frac{(1/L)^2}{(1/L)^4} \cdot \int_{-\infty}^{\infty} dp \cdot \frac{1}{1 + \left[\frac{(P_0 - P)L}{\hbar}\right]^2} \cdot p$$

$$= \frac{2L}{\pi\hbar} \cdot \int_{-\infty}^{\infty} dp\ \frac{1}{1 + \left(\frac{PL}{\hbar}\right)^2} \cdot (P + P_0) \qquad \boxed{\begin{array}{c} P \to P + P_0 \\ (\text{shift}) \end{array}}$$

$$= \frac{2L}{\pi\hbar} \int_{-\infty}^{\infty} dp \left[\underbrace{\cancel{\frac{1}{1 + \left(\frac{PL}{\hbar}\right)^2} \cdot p}}_{\text{odd}} + \frac{1}{1 + \left(\frac{PL}{\hbar}\right)^2} \cdot P_0 \right]$$

$$= \frac{2L}{\pi\hbar} \cdot P_0 \cdot \int_{-\infty}^{\infty} dp \left[\frac{1}{1 + \left(\frac{PL}{\hbar}\right)^2} \right]^2 \qquad \boxed{k = \frac{PL}{\hbar},\ dk = \frac{L}{\hbar} dp}$$

$$= \frac{2L}{\pi\hbar} P_0 \cdot \frac{\hbar}{L} \cdot \underbrace{\int_{-\infty}^{\infty} dk \left[\frac{1}{1 + k^2} \right]^2}_{= \pi/2}$$

$$= P_0 .$$

$$\therefore \underline{\langle p \rangle = P_0}$$

$$\langle p^2 \rangle = \int dp \;\phi^*(p)\; p^2 \;\phi(p)$$

$$= \frac{2}{\pi\hbar L} \int_{-\infty}^{\infty} dp \left[\frac{(1/L)}{(1/L)^2 + (p_0 - p/\hbar)^2} \right]^2 p^2$$

$$= \frac{2}{\pi\hbar L} \cdot L^2 \int_{-\infty}^{\infty} dp \left[\frac{1}{1 + \left(\frac{(p_0 - p)L}{\hbar}\right)^2} \right]^2 p^2$$

shift

$$= \frac{2L}{\pi\hbar} \int_{-\infty}^{\infty} dp \left[\frac{1}{1 + \left(\frac{L}{\hbar}p\right)^2} \right]^2 (p + p_0)^2$$

odd 삭제

$$= \frac{2L}{\pi\hbar} \times \int_{-\infty}^{\infty} dp \cdot \left[\frac{1}{1 + \left(\frac{L}{\hbar}p\right)^2} \right]^2 (p^2 + p_0^2)$$

$$= \frac{2L}{\pi\hbar} \int_{-\infty}^{\infty} dp \cdot \left[\frac{1}{1 + \left(\frac{L}{\hbar}p\right)^2} \right]^2 p^2 \; + \; p_0^2$$

$$k = \frac{L}{\hbar}p \qquad dk = \frac{L}{\hbar}dp$$

$$= \frac{2L}{\pi\hbar} \int_{-\infty}^{\infty} \frac{\hbar}{L} dk \cdot \left[\frac{1}{1 + k^2} \right]^2 \left(\frac{\hbar}{L}\right)^2 \cdot k^2 \; + p_0^2$$

$$= \frac{2}{\pi} \cdot \left(\frac{\hbar}{L}\right)^2 \int_{-\infty}^{\infty} dk \cdot \left(\frac{1}{1+k^2}\right)^2 \cdot k^2 \; + p_0^2$$

$k = \tan\theta$ 로 치환. $dk = \sec^2\theta \, d\theta$.

$$\therefore \langle p^2 \rangle = \frac{2}{\pi}\left(\frac{\hbar}{L}\right)^2 \cdot \int_{-\pi/2}^{\pi/2} d\theta \cdot \sec^2\theta \cdot \frac{1}{\sec^4\theta} \cdot \tan^2\theta + P_0^2$$

$$= \frac{2}{\pi}\left(\frac{\hbar}{L}\right)^2 \cdot \int_{-\pi/2}^{\pi/2} d\theta \cdot \sin^2\theta + P_0^2$$

$$= \left(\frac{\hbar}{L}\right)^2 + P_0^2 .$$

v) 정리.

$\langle x \rangle = 0 . \quad \langle x^2 \rangle = \frac{L^2}{2} \rightarrow \Delta x = \frac{L}{\sqrt{2}}$

$\langle p \rangle = P_0 \quad \langle p^2 \rangle = \left(\frac{\hbar}{L}\right)^2 + P_0^2 \rightarrow \Delta p = \frac{\hbar}{L}.$

vi) uncertainty relation.

$$\Delta x \cdot \Delta p = \frac{L}{\sqrt{2}} \cdot \frac{\hbar}{L} = \frac{\hbar}{\sqrt{2}} > \frac{\hbar}{2} .$$

satisfied!

4. Consider a state function which is real,

$\psi(x) = \psi^*(x)$.

(a) Show that $\langle p \rangle = 0$. What about $\langle p^2 \rangle$ & $\langle x \rangle$?

⇒ i) $\langle p \rangle = \int dx\, \psi^*(x) \left(\frac{\hbar}{i}\frac{d}{dx}\right) \psi(x)$

이때.. $\langle p \rangle$ 는 physical observable 의 expectation value 이므로.. real 이어야 한다!

$\therefore \langle p \rangle = \langle p \rangle^*$

$\therefore \langle p \rangle^* = \int dx\, \psi(x) \left(-\frac{\hbar}{i}\frac{d}{dx}\right) \psi^*(x)$

이때.. $\psi(x)$ 가 real 이라면..

$\langle p \rangle = \int dx\, \psi(x) \left(\frac{\hbar}{i}\frac{d}{dx}\right) \psi(x)$

$\langle p \rangle^* = -\int dx\, \psi(x) \left(\frac{\hbar}{i}\frac{d}{dx}\right) \psi(x)$.

서로 동일한 양이 $(\langle p \rangle = \langle p \rangle^*)$, 서로 다른 부호를 가지고 그 절대값이 같으므로.. $\langle p \rangle = 0$ 이다.

ii) $\langle p^2 \rangle = \int dx\, \psi^*(x) \left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 \psi(x)$

$= \int dx\, \psi(x) \left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 \psi(x)$ ← ψ가 real

ψ가 real 이라는 조건만으로는.. $\langle p^2 \rangle$가 0인지 결정할 수 없다!

ex) 1번에서 고려하였던

$\psi(x) = A\exp(-ax^2)$ 은 real 이나.

$\langle p^2 \rangle = \hbar^2 a \neq 0$ 이였다.

iii) $\langle x \rangle$ 역시.. ψ가 real 이라는 조건으로 0인지 아닌지 판단할 수 없다!

ex) $\psi(x) = A\exp\left[-a(x-x_0)^2\right]$ $(x_0 \neq 0)$.

같은 경우.. real 이나 $\langle x \rangle = x_0 \neq 0$ 이다.

(b) Under what conditions of $\psi(x)$, is $\phi(p)$ is real, and what then is $\langle x \rangle$?

⇒ i) $\phi(p) = \int dx \frac{1}{\sqrt{2\pi\hbar}} e^{-i\frac{p}{\hbar}x} \psi(x)$.

$\downarrow$

$\phi^*(p) = \int dx \frac{1}{\sqrt{2\pi\hbar}} e^{+i\frac{p}{\hbar}x} \psi^*(x)$.

ii) $\phi(p)$가 real 이라면..

$\phi(p) = \phi^*(p)$ 이고.. 따라서

$$\int dx \frac{1}{\sqrt{2\pi\hbar}} e^{-i\frac{p}{\hbar}x} \psi(x)$$

$$= \int dx \frac{1}{\sqrt{2\pi\hbar}} e^{+i\frac{p}{\hbar}x} \psi^*(x)$$ 이다.

여기 우변을 좀더 살펴보면..

$$\int dx \frac{1}{\sqrt{2\pi\hbar}} e^{+i\frac{p}{\hbar}x} \psi^*(x) \quad \Big) \; x \to -x$$

$$= \int dx \frac{1}{\sqrt{2\pi\hbar}} e^{-i\frac{p}{\hbar}x} \psi^*(-x).$$

$\boxed{\therefore \psi(x) = \psi^*(-x) \text{ 이면.. } \phi(p) \text{는 real!}}$

iii) 만약 $\psi(x) = \psi^*(-x)$ 라면..

$$\langle x \rangle = \int dx\ \psi^*(x)\ x\ \psi(x)$$

$$= \int dx\ x \cdot \psi^*(x)\psi(x)$$ 이다.

이때 $\psi^*(x)\psi(x)$ 의 parity 를 살펴보자.

$$\psi^*(-x)\psi(-x) = \psi(x) \cdot \psi^*(x).$$

$\therefore$ parity even!

$\therefore \langle x \rangle = 0$ ($\because$ parity odd function의 적분이므로.)

5. In class, I argued that the commutation relation $[x, p] = i\hbar$ is fundamental, and the form of the operators x and p in coordinate, and momentum spaces are devised to satisfy the commutation relation. For example, in coordinate space, we put $x_{op} = x$, and $p_{op} = -i\hbar\,\partial/\partial x$.

Let us define, in coordinate space, $x_{op} = x + x_0$, $p_{op} = -i\hbar\,\partial/\partial x + p_0$, where x_0 and p_0 are real numbers. These operators also satisfy the above commutation relation. Then what is the meaning of these new operators?

And what are the forms of the newly defined operators in momentum space?

⇒ i) $x_{op} = x + x_0$, $P_{op} = -i\hbar\,\partial/\partial x + P_0$ 로 새롭게 정의하면..

임의의 state $|\alpha\rangle$ 에 대해.. x_{op} 와 P_{op} 의

expectation value 가 x_0, P_0 만큼 각각 shift하게 된다.

$$\langle x_{op}\rangle = \langle\alpha|x_{op}|\alpha\rangle = \langle\alpha|x|\alpha\rangle + \langle\alpha|x_0|\alpha\rangle$$

$$= \langle x\rangle_{original} + x_0 \underbrace{\langle\alpha|\alpha\rangle}_{1}$$

$$= \langle x\rangle_{original} + x_0$$

$$\langle P_{op}\rangle = \langle P\rangle_{original} + P_0.$$

이는 결국 momentum 과 position 의 눈금을 옮긴 것에 불과하다.

ii) configuration space 에서 새롭게 정의된 x_{op} , P_{op}는 momentum space 에서는 어떻게 보일까?

어떠한 basis 를 쓰건 간에.. physical observable 의 expectation value는 동일해야 하므로.. momentum space 에서 P_{op}는 다음과 같이 표현된다.

$$P_{op} = P + P_0 .$$

마찬가지로.. x_{op} 역시 다음과 같이 표현되어야 한다.

$$x_{op} = +i\hbar\, \partial/\partial p .$$