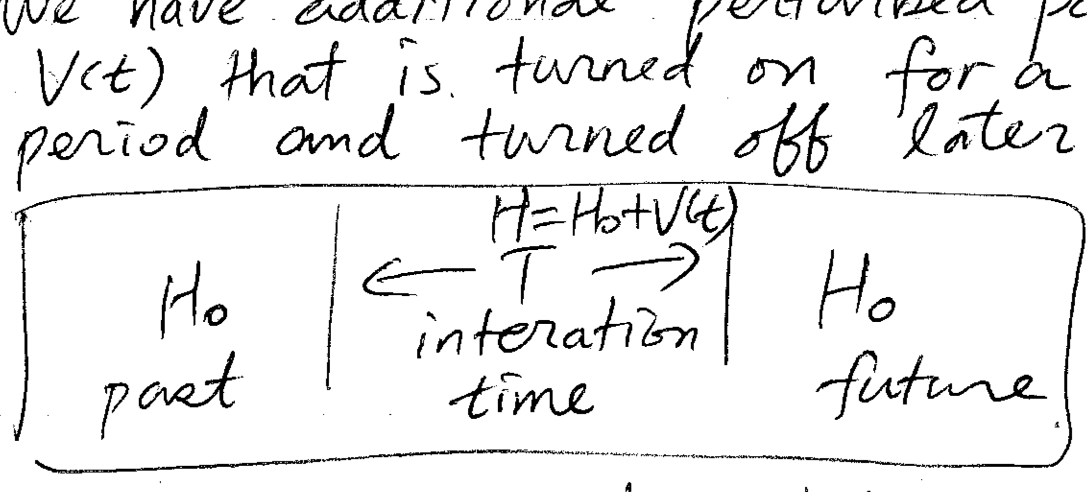
5.5 Time-dependent perturbation theory

H_0 unperturbed Hamiltonian whose eigenvalue problem is solvable:

$$H_0|n\rangle = E_n|n\rangle.$$

We have additional perturbed potential $V(t)$ that is turned on for a period and turned off later



Suppose $|\alpha(t)\rangle$ is the solution of

$$H|\alpha(t)\rangle = E|\alpha(t)\rangle$$

At $t=0$, $H = H_0$.

$$\therefore \; H_0|\alpha(t)\rangle\Big|_{t=0} = E_n|\alpha(0)\rangle$$

$$|\alpha(t)\rangle\Big|_{t=0} = \sum_n c_n(0)|n\rangle$$

If $V(t)=0$, then

$$|\alpha(t)\rangle\Big|_{V=0} = \sum_n c_n(0)\, e^{-\frac{i}{\hbar}E_n t}|n\rangle$$

If $V(t)\neq 0$, then c_n ~~must~~ may acquire time dependence.

$$|\alpha(t)\rangle = \sum_n c_n(t)\, e^{-\frac{i}{\hbar}E_n t}|n\rangle$$

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① $c_n(0)$ can be determined if we know $|\alpha(t=0)\rangle$.

② $c_n(t)$ can be solved:

$$i\hbar\frac{\partial}{\partial t}|\alpha(t)\rangle = [H_0 + V(t)]\,|\alpha(t)\rangle$$

The interaction picture.

In the Schrödinger picture

$$|\alpha(t)\rangle\Big|_{V=0} = e^{-\frac{i}{\hbar}H_0 t}|\alpha(0)\rangle.$$

In the interaction picture, the time evolution effect due to H_0 is removed. It is useful because we are interested in the change in $c_n(t)$.

$$|\alpha(t)\rangle_I = e^{\frac{i}{\hbar}H_0 t}|\alpha(t)\rangle_S \leftarrow \text{Schrödinger}$$

In the interaction picture, an operator of the Schrödinger picture becomes

$$A_I(t) = e^{\frac{i}{\hbar}H_0 t} A_S\, e^{-\frac{i}{\hbar}H_0 t}.$$

$$V_I(t) = e^{\frac{i}{\hbar}H_0 t} V_S(t)\, e^{-\frac{i}{\hbar}H_0 t}.$$

We recall the Heisenberg picture

$$|\alpha\rangle_H = e^{+\frac{i}{\hbar}Ht}|\alpha(t)\rangle_S ,$$

$$A_H(t) = e^{\frac{i}{\hbar}Ht} A_S \, e^{-\frac{i}{\hbar}Ht} .$$

Let us investigate the Schrödinger equation in the interaction picture.

First, in the Schrödinger picture,

$$i\hbar\frac{\partial}{\partial t}|\alpha(t)\rangle_S = [H_0 + V(t)]\,|\alpha(t)\rangle_S$$

$$|\alpha(t)\rangle_S = e^{-\frac{i}{\hbar}H_0 t}\, e^{\frac{i}{\hbar}H_0 t}\,|\alpha(t)\rangle_S$$

$$= e^{-\frac{i}{\hbar}H_0 t}|\alpha(t)\rangle_I$$

$$\therefore\ i\hbar\frac{\partial}{\partial t}\left[e^{-\frac{i}{\hbar}H_0 t}|\alpha(t)\rangle_I\right] = \left[H_0 + V(t)\right]\left[e^{-\frac{i}{\hbar}H_0 t}|\alpha(t)\rangle_I\right]$$

multiplying $e^{\frac{i}{\hbar}H_0 t}$ to the left,

we find that

$$e^{\frac{i}{\hbar}H_0 t}\, i\hbar\frac{\partial}{\partial t}\left[e^{-\frac{i}{\hbar}H_0 t}|\alpha(t)\rangle_I\right]$$

$$= e^{\frac{i}{\hbar}H_0 t}\left[H_0 + V(t)\right] e^{-\frac{i}{\hbar}H_0 t}\,|\alpha(t)\rangle_I$$

Note that $i\hbar\frac{\partial}{\partial t}\, e^{-\frac{i}{\hbar}H_0 t}|\alpha(t)\rangle_I$

$$= e^{-\frac{i}{\hbar}H_0 t}\left[H_0 + i\hbar\frac{\partial}{\partial t}\right]|\alpha(t)\rangle_I$$

$$\therefore\ LHS = \left[H_0 + i\hbar\frac{\partial}{\partial t}\right]|\alpha(t)\rangle_I$$

$$LHS = \left[H_0 + i\hbar \frac{\partial}{\partial t}\right] |\alpha(t)\rangle_I$$

$$RHS = \left[H_0 + V_I(t)\right] |\alpha(t)\rangle_I$$

Therefore, the Schrödinger equation is simplified as

$$i\hbar \frac{\partial}{\partial t} |\alpha(t)\rangle_I = V_I(t) |\alpha(t)\rangle_I .$$

We observe that the contribution of H_0 cancels.

In the Heisenberg picture

$$A_H(t) = e^{\frac{i}{\hbar}Ht} A_S \, e^{-\frac{i}{\hbar}Ht}$$

$$[A_H, H] = e^{\frac{i}{\hbar}Ht} [A_S, H] \, e^{-\frac{i}{\hbar}Ht}$$

$$= \left[e^{\frac{i}{\hbar}Ht} A_S \, e^{-\frac{i}{\hbar}Ht}, H\right]$$

$$\frac{d}{dt} A_H = \frac{i}{\hbar} e^{\frac{i}{\hbar}Ht} [H, A_S] \, e^{-\frac{i}{\hbar}Ht}$$

$$= \frac{1}{i\hbar} [A_H, H]$$

We can find the analogy to the interaction picture.

$$A_I = e^{\frac{i}{\hbar}H_0 t} A_S \, e^{-\frac{i}{\hbar}H_0 t} \Leftarrow \text{(If } A_S \text{ is independent of } t\text{)}$$

$$\frac{dA_I}{dt} = \frac{i}{\hbar} e^{\frac{i}{\hbar}H_0 t} [H_0, A_S] e^{-\frac{i}{\hbar}H_0 t}$$

$$= \frac{1}{i\hbar}[A_I(t), H_0].$$

We return to the Schrödinger equation in the interaction picture:

$$\boxed{i\hbar \frac{\partial}{\partial t} |\alpha(t)\rangle_I = V_I(t) |\alpha(t)\rangle_I}$$

Substituting ~~$|\alpha(t)\rangle_I = \sum_n c_n(t)$~~

$$|\alpha(t)\rangle_S = \sum_n c_n(t) \, e^{-\frac{i}{\hbar}E_n t} |n\rangle.$$

In the interaction picture,

$$|\alpha(t)\rangle_I = e^{+\frac{i}{\hbar}H_0 t} \sum_n c_n(t) \, e^{-\frac{i}{\hbar}E_n t} |n\rangle$$

Note that $e^{\frac{i}{\hbar}H_0 t}|n\rangle = e^{\frac{i}{\hbar}E_n t}|n\rangle$

$$= \sum_n c_n(t) |n\rangle.$$

In the interaction picture, the time-evolution factor $e^{-\frac{i}{\hbar}E_n t}$ disappears.

Substituting $|\alpha(t)\rangle_I = \sum_n c_n(t)\,|n\rangle$,

into $i\hbar \frac{\partial}{\partial t}|\alpha(t)\rangle_I = V_I(t)\,|\alpha(t)\rangle_I$

$$i\hbar \frac{\partial}{\partial t}\sum_n c_n(t)\,|n\rangle = V_I(t)\sum_m c_m(t)\,|m\rangle$$

Multiplying $\langle m|$ to the left, we find that

$$\sum_n i\hbar \frac{dc_n(t)}{dt}\underbrace{\langle m|n\rangle}_{\delta_{mn}} = \sum_n \langle m|\,V_I(t)\,|n\rangle\, c_n(t)$$

$$\therefore\ i\hbar \frac{dc_m(t)}{dt} = \sum_n \langle m|\,V_I(t)\,|n\rangle\, c_n(t)$$

If we employ the matrix representation, then we have

$$i\hbar\begin{pmatrix}\dot{c}_1(t)\\ \dot{c}_2(t)\\ \vdots\end{pmatrix} = \begin{pmatrix} & & \\ & V_I(t) & \\ & & \end{pmatrix}\begin{pmatrix}c_1(t)\\ c_2(t)\\ \vdots\end{pmatrix}$$

Note that the matrix element $\langle m|V_I(t)|n\rangle$ is

$$\langle m|V_I(t)|n\rangle = \langle m|\, e^{+\frac{i}{\hbar}H_0 t}\, V_S(t)\, e^{-\frac{i}{\hbar}H_0 t}\,|n\rangle$$

$$= e^{\frac{i}{\hbar}(E_m-E_n)t}\,\langle m|\,V_S(t)\,|n\rangle$$

$$= e^{i\omega_{mn}t}\,\langle m|\,V_S(t)\,|n\rangle$$

$\omega_{mn} = \frac{E_m - E_n}{\hbar}$.

Time-dependent two-state problem.

The unperturbed Hamiltonian H_0 is given by

$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \rightarrow E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2| .$$

The interaction Hamiltonian is

$$V(t) = \begin{pmatrix} 0 & \gamma e^{i\omega t} \\ \gamma e^{-i\omega t} & 0 \end{pmatrix} \rightarrow \gamma\left[e^{i\omega t}|1\rangle\langle 2| + e^{-i\omega t}|2\rangle\langle 1|\right],$$

where γ and ω are real constants.

Therefore $V^{\dagger}(t) = V(t)$.

We assume that

$$|\alpha(t=0)\rangle = |1\rangle$$

We assume $E_2 > E_1$

Therefore $c_1(0) = 1$ and $c_2(0) = 0$.

In the interaction picture, the Schrödinger equation becomes

$$i\hbar \begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \end{pmatrix} = \begin{pmatrix} 0 & \gamma e^{i\omega t} e^{-i\omega_{21} t} \\ \gamma e^{-i\omega t} e^{+i\omega_{21} t} & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\omega_{21} = \frac{E_2 - E_1}{\hbar} > 0$$

$$= \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \qquad a = \gamma e^{i(\omega + \omega_{12})t}$$

$$a^* = \gamma e^{-i(\omega + \omega_{12})t}$$

$$\dot{a}^* = -i(\omega-\omega_{21})a^*$$

$$i\hbar\dot{c}_1 = \gamma a c_2 \qquad a = e^{i(\omega-\omega_{21})t}$$

$$i\hbar\dot{c}_2 = \gamma a^* c_1 \qquad a^* = e^{-i(\omega-\omega_{21})t}$$

$$aa^* = 1$$

$$i\hbar\ddot{c}_2 = \gamma\dot{a}^* c_1 + \gamma a^*\dot{c}_1$$

$$\dot{c}_1 = \frac{\gamma a}{i\hbar}c_2$$

$$i\hbar\ddot{c}_2 = \gamma\left[-i(\omega-\omega_{21})\right]a^* c_1 + \gamma^2(a^* a)\frac{c_2}{i\hbar}$$

$$\gamma a^* c_1 = i\hbar\dot{c}_2$$

$$\therefore\ i\hbar\ddot{c}_2 = \hbar(\omega-\omega_{21})\dot{c}_2 + \gamma^2\frac{c_2}{i\hbar}$$

Multiplying "$+\frac{i}{\hbar}$" to the equation,

$$-\ddot{c}_2 = i(\omega-\omega_{21})\dot{c}_2 + \frac{\gamma^2}{\hbar^2}c_2$$

$$\Rightarrow \boxed{\ddot{c}_2 + i(\omega-\omega_{21})\dot{c}_2 + \frac{\gamma^2}{\hbar^2}c_2 = 0}$$

Substituting the trial solution

$$c_2(t) = e^{\lambda t}$$

$$\lambda^2 + i(\omega-\omega_{21})\lambda + \frac{\gamma^2}{\hbar^2} = 0$$

$$\lambda^2 + i(\omega-\omega_{21})\lambda + \frac{\gamma^2}{\hbar^2} = 0$$

$$\left[\lambda + \frac{i}{2}(\omega-\omega_{21})\right]^2 + \left(\frac{\omega-\omega_{21}}{2}\right)^2 + \frac{\gamma^2}{\hbar^2} = 0$$

$$\lambda + \frac{i}{2}(\omega-\omega_{21}) = \pm i\sqrt{\left(\frac{\omega-\omega_{21}}{2}\right)^2 + \left(\frac{\gamma}{\hbar}\right)^2}$$

$$\therefore \lambda = i\left[-\left(\frac{\omega-\omega_{21}}{2}\right) \pm \sqrt{\left(\frac{\omega-\omega_{21}}{2}\right)^2 + \left(\frac{\gamma}{\hbar}\right)^2}\right]$$

The solution is

$$c_2(t) = e^{-i\left(\frac{\omega-\omega_{21}}{2}\right)t}\left[\alpha\cos\left(\sqrt{\left(\frac{\omega-\omega_{21}}{2}\right)^2 + \left(\frac{\gamma}{\hbar}\right)^2}\,t\right) + \beta\sin\left(\sqrt{\left(\frac{\omega-\omega_{21}}{2}\right)^2 + \left(\frac{\gamma}{\hbar}\right)^2}\,t\right)\right].$$

At $t=0$

$$c_2(0) = \alpha + \beta\cdot 0 = 0 \quad \therefore \alpha = 0$$

$$c_2(t) = e^{-i\left(\frac{\omega-\omega_{21}}{2}\right)t}\beta\sin\left(\sqrt{\left(\frac{\omega-\omega_{12}}{2}\right)^2 + \left(\frac{\gamma}{\hbar}\right)^2}\,t\right)$$

$$i\hbar\dot{c}_2(t) = \beta\frac{\hbar(\omega-\omega_{21})}{2}e^{-i\left(\frac{\omega-\omega_{21}}{2}\right)t}\sin\sqrt{\quad''\quad}\,t$$

$$+ i\hbar\beta\sqrt{(\quad)^2+(\quad)^2}\,e^{-i\left(\frac{\omega-\omega_{21}}{2}\right)t}\cos\sqrt{\quad}\,t$$

$$i\hbar\dot{c}_2(0) = \gamma c_1(0) = \gamma$$

$$\therefore \quad \beta = \frac{-i\gamma/\hbar}{\sqrt{\left(\frac{\omega-\omega_{21}}{2}\right)^2 + \left(\frac{\gamma}{\hbar}\right)^2}}$$

$$\therefore\ C_2(t) = \frac{-i\gamma/\hbar}{\sqrt{\left(\frac{\omega-\omega_{21}}{2}\right)^2+\left(\frac{\gamma}{\hbar}\right)^2}}\, e^{-i\left(\frac{\omega-\omega_{21}}{2}\right)t} \sin\left[\sqrt{\left(\frac{\omega-\omega_{21}}{2}\right)^2+\left(\frac{\gamma}{\hbar}\right)^2}\, t\right]$$

Therefore,

$$|C_2(t)|^2 = \frac{\left(\frac{\gamma}{\hbar}\right)^2}{\left(\frac{\omega-\omega_{21}}{2}\right)^2+\left(\frac{\gamma}{\hbar}\right)^2} \sin^2\left[\sqrt{\left(\frac{\omega-\omega_{21}}{2}\right)^2+\left(\frac{\gamma}{\hbar}\right)^2}\, t\right].$$

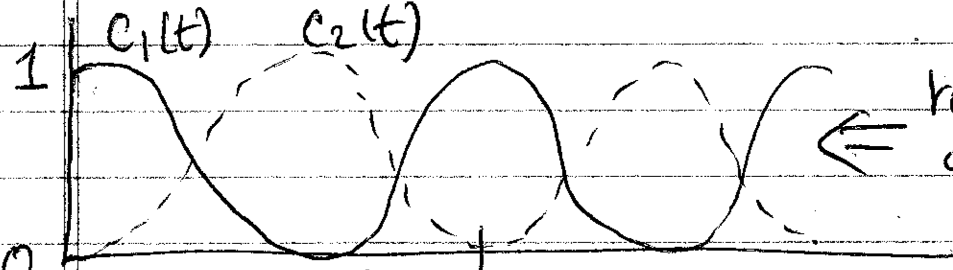
← Rabi's formula

$$|C_1(t)|^2 = |C_1(t)|^2 + |C_2(t)|^2 - |C_2(t)|^2$$
$$= 1 - |C_2(t)|^2.$$

Note that $|C_1(t)|^2 + |C_2(t)|^2 = 1$

Because $V = V^\dagger$.

$$\Omega \equiv \sqrt{\left(\frac{\omega-\omega_{21}}{2}\right)^2+\left(\frac{\gamma}{\hbar}\right)^2}$$ is called Rabi frequency.



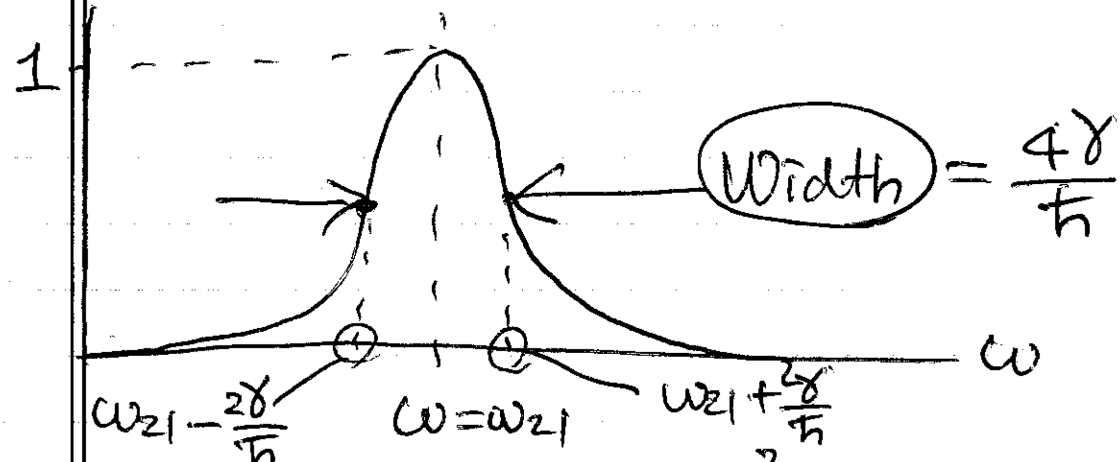
⇐ repeated absorption and emission (cyclic)

$$T = \frac{2\pi}{\sqrt{\left(\frac{\omega-\omega_{21}}{2}\right)^2+\left(\frac{\gamma}{\hbar}\right)^2}}$$

resonance condition $\omega = \omega_{21}$; $\Rightarrow \Omega = \frac{\gamma}{\hbar}$

When $\omega = \omega_{21}$, ~~the~~ it becomes the resonance.

$$|c_2^{max}(t)|^2 = \frac{(\gamma/\hbar)^2}{\left(\frac{\omega-\omega_{21}}{2}\right)^2 + \left(\frac{\gamma}{\hbar}\right)^2} = \frac{1}{1+\left(\frac{\hbar(\omega-\omega_{21})}{2\gamma}\right)^2}$$



If we scale $|c_2^{max}(t)|^2 = 1$ over the whole range of ω, the value is obtained at the resonance: $\omega = \omega_{21}$.

When $|c_2^{max}(t)|^2 = \frac{1}{2}$, $\left(\frac{\hbar(\omega-\omega_{21})}{2\gamma}\right)^2 = 1$

$$\therefore \omega = \omega_{21} \pm \frac{2\gamma}{\hbar}.$$

The width of the resonance in the frequency domain is defined by the full width at half maximum of $|c_2^{max}(t)|^2$ that is $\frac{4\gamma}{\hbar}$.

(Homework)
As applications, study Spin magnetic resonance and Maser in the remainder of Section 5.5 of Sakurai.