Communication Systems II

[KECE322_01]

<2012-2nd Semester>

Lecture #8

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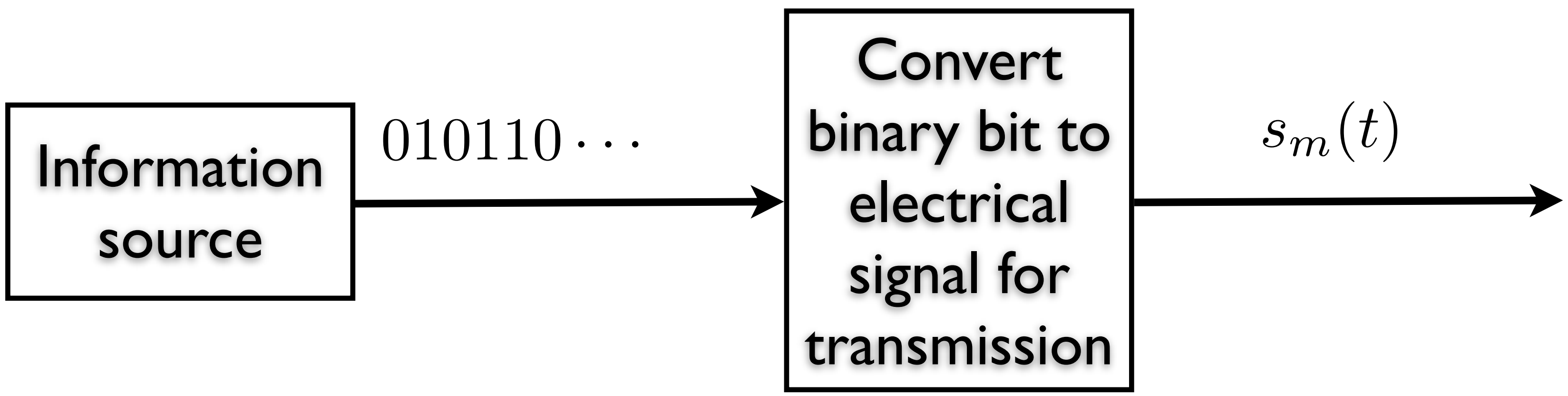
Outline

- Binary pulse modulation
  - Binary pulse amplitude modulation
  - Binary pulse position modulation
- Geometric representation of signal waveform
- Optimum receiver over AWGN

Digital Modulation

- Digital modulation



  - Converting the binary bit (or bits) to electrical signal for transmission is called "digital modulation".

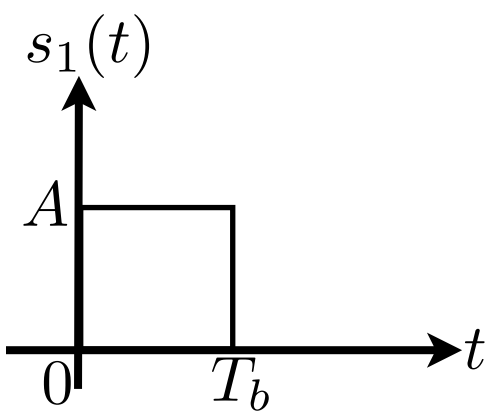
- Carrier modulation
  - If we upconvert $s_m(t)$ so that its power resided in high frequency area, it is called carrier modulation.
  - Carrier modulation can be possible by multiplying $\cos(2\pi f_c t)$ (or $\sin(2\pi f_c t)$) with high value of f_c to $s_m(t)$.

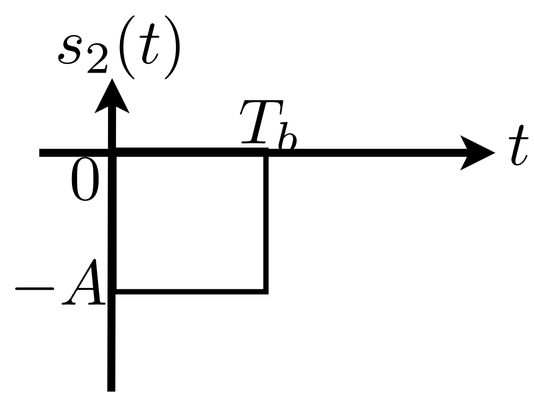
Binary vs. M-ary Modulation

- Binary modulation
  - If one bit is mapped to a signal, it is called "binary modulation".
  - In this case, there are two possible signals, $s_1(t)$ and $s_2(t)$.
- M-ary modulation
  - If M bits are mapped to a signal, it is called "M-ary modulation".
  - In this case, there are 2^M possible signals, $s_1(t), s_2(t). \ldots, s_{2^M}(t)$.

Binary Pulse Amplitude Modulation (PAM)

- Signal waveform



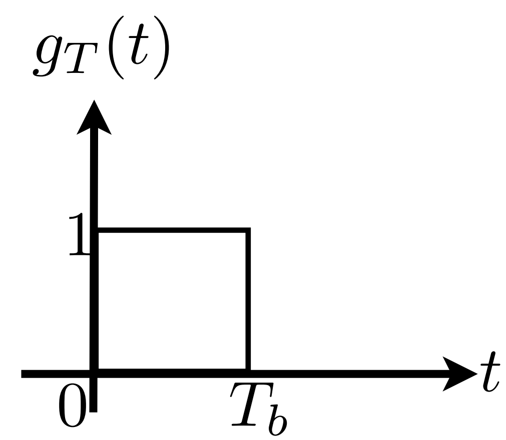


T_b : bit interval

$$s_m(t) = A_m g_T(t), \qquad 0 \le t \le T_b, \;\; m = 1, 2$$

$$A_m = A, \;\; (\text{if } m = 1)$$

$$A_m = -A, \;\; (\text{if } m = 2)$$



- Bit rate

$$R_b = \frac{1}{T_b} \;\; \text{bits/sec}$$

Signal energy

$$\begin{aligned} \mathcal{E}_m &= \int_0^{T_b} s_m^2(t)\, dt, \;\; m = 1, 2 \\ &= A^2 \int_0^{T_b} g_T^2(t)\, dt \\ &= A^2 T_b \end{aligned}$$

- The two signal waveforms have equal energy, i.e., $\mathcal{E}_m = A^2 T_b, \text{ for } m = 1, 2.$

- Define the *signal energy per bit* as $\mathcal{E}_b$

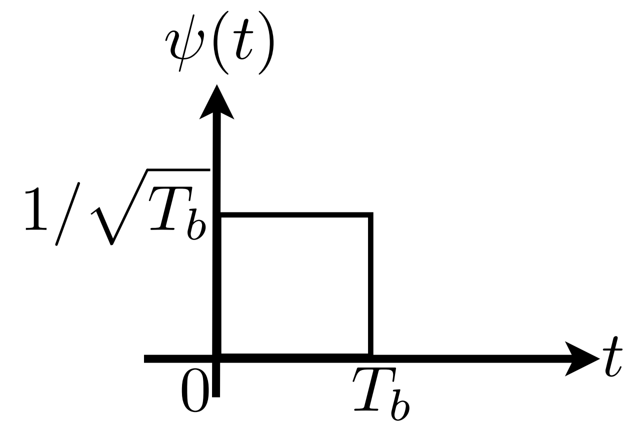
$$\mathcal{E}_b = A^2 T_b \implies A = \sqrt{\frac{\mathcal{E}_b}{T_b}}$$

Geometric representation

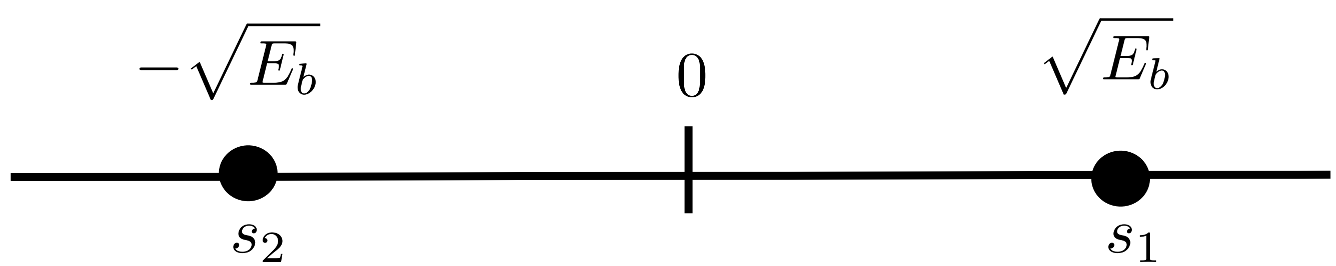
$$s_m(t) = s_m \psi(t), \quad m = 1, 2$$

where

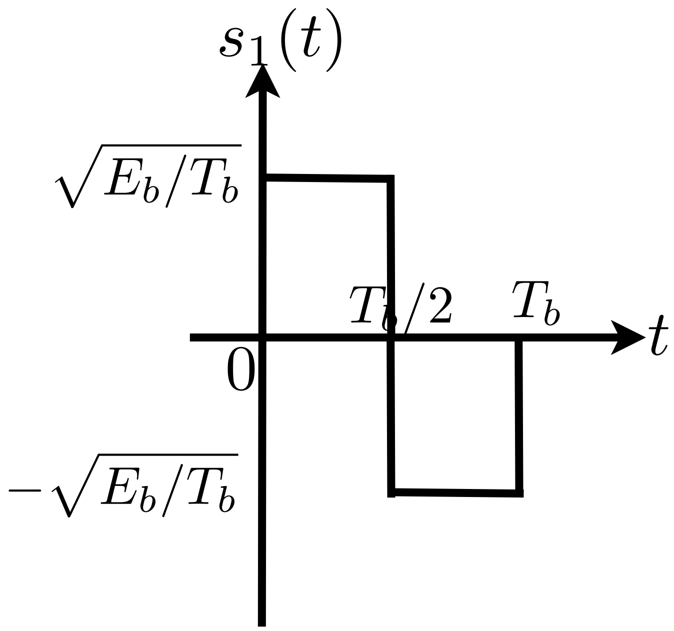
$$s_1 = \sqrt{E_b}, \quad s_2 = -\sqrt{E_b}$$

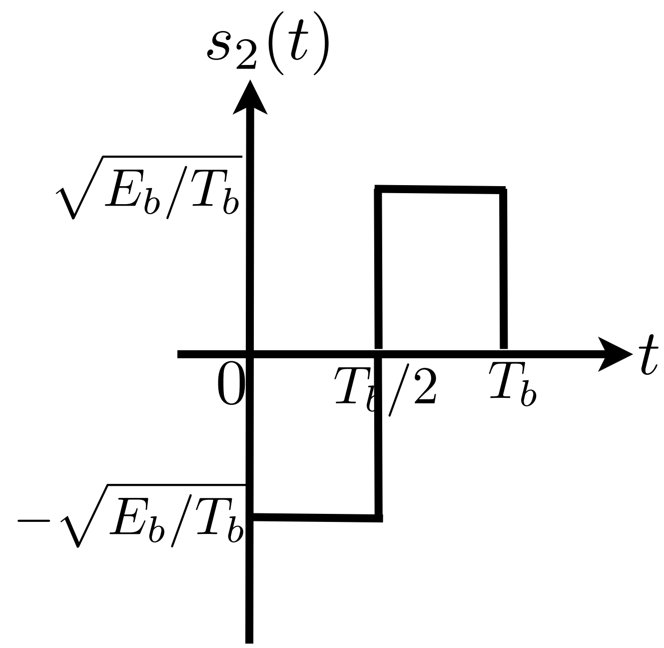


- Signal constellation (or space diagram) based on geometric representation



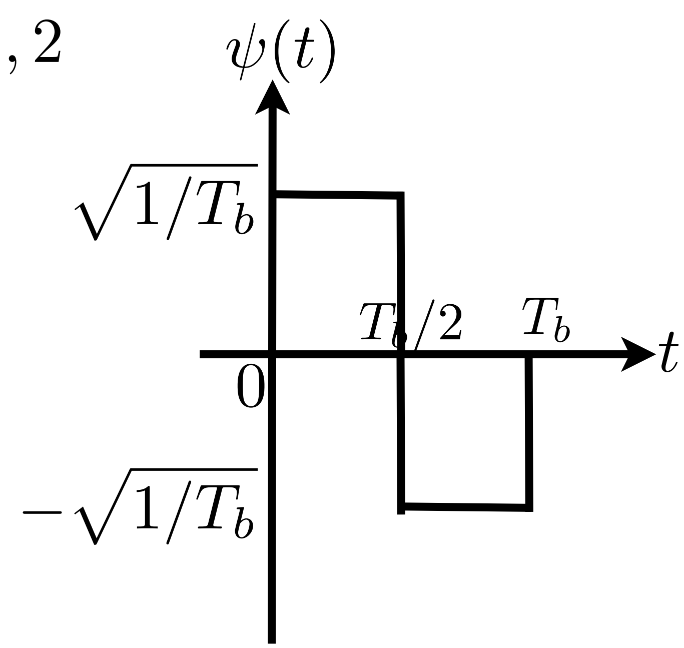
- Example of binary antipodal signal





$$s_m(t) = s_m\psi(t), \quad m = 1, 2$$

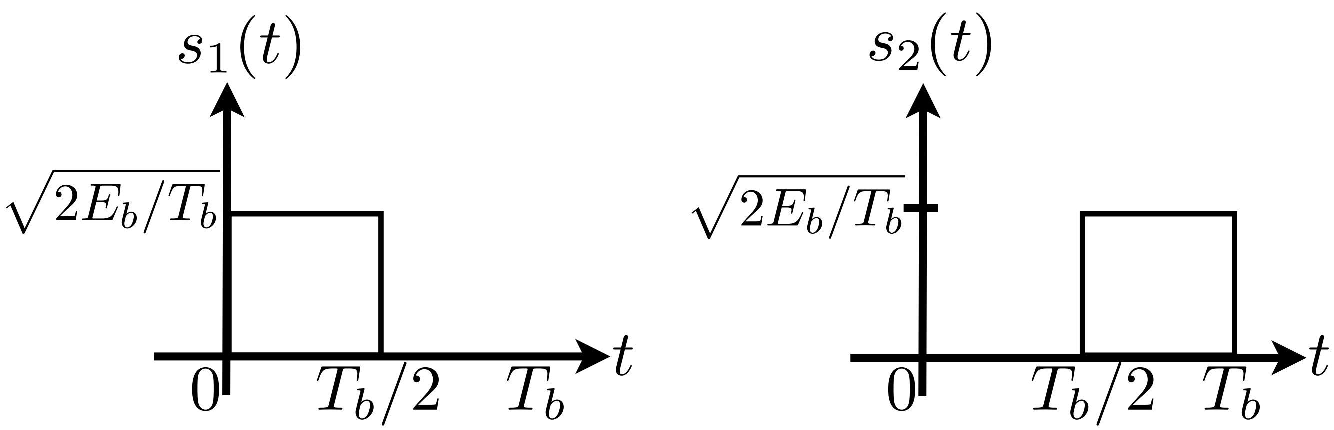
$$s_1 = \sqrt{E_b}, \quad s_2 = -\sqrt{E_b}$$



- Any antipodal signal waveforms can be represented geometrically as two vectors (two signal points) on the real line, where one vector is the negative of the other.

Binary Pulse Position Modulation (PPM)

- Signal waveform



  - PPM signals are *orthogonal*, i,e.,

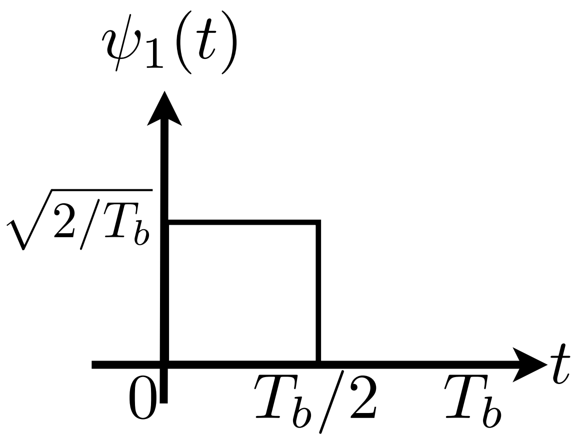
$$\int_0^{T_b} s_1(t)s_2(t)\,dt = 0$$

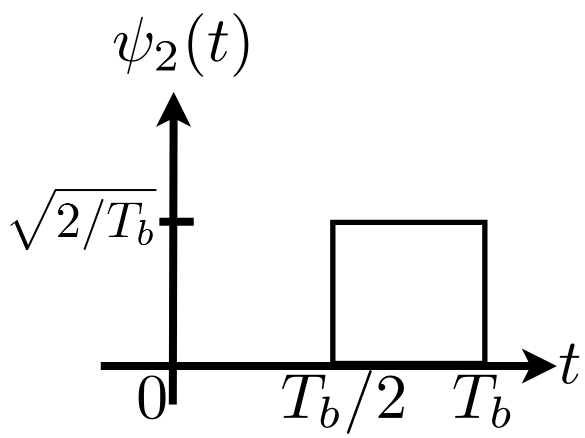
- Energy

$$\mathcal{E}_b = \int_0^{T_b} s_1^2(t)\,dt = \int_0^{T_b} s_2^2(t)\,dt$$

Geometric representation

$$
\begin{aligned}
s_1(t) &= s_{11}\psi_1(t) + s_{12}\psi_2(t) \\
s_2(t) &= s_{21}\psi_1(t) + s_{22}\psi_2(t)
\end{aligned}
$$

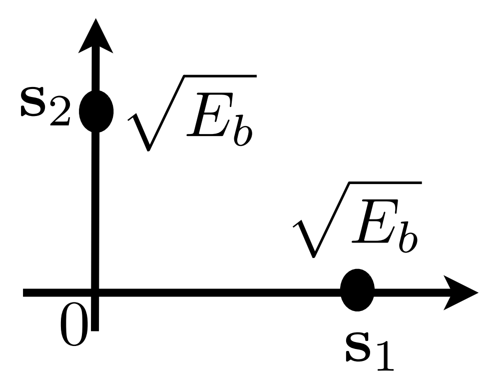




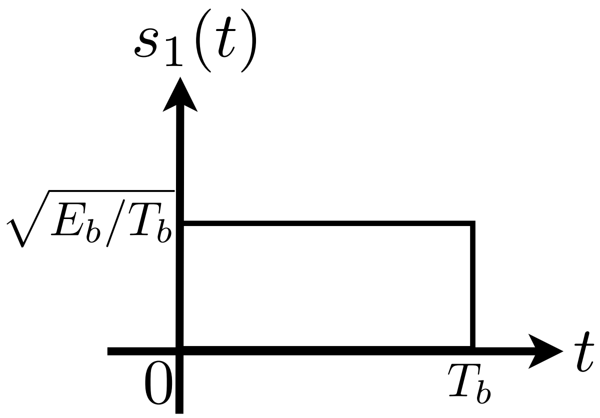
$$
\begin{aligned}
s_{11} &= \int_0^{T_b} s_1(t)\psi_1(t)\,dt = \sqrt{E_b} \\
s_{12} &= \int_0^{T_b} s_1(t)\psi_2(t)\,dt = 0 \\
s_{21} &= \int_0^{T_b} s_2(t)\psi_1(t)\,dt = 0 \\
s_{22} &= \int_0^{T_b} s_2(t)\psi_2(t)\,dt = \sqrt{E_b}
\end{aligned}
$$

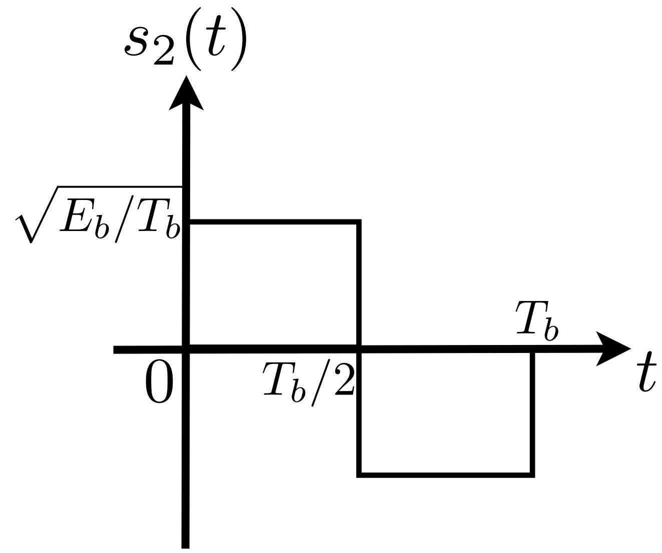
- In this case, the two signal waveforms are represented as two-dimensional vectors

$$
\begin{aligned}
\mathbf{s}_1 &= (s_{11}, 0) = (\sqrt{E_b}, 0) \\
\mathbf{s}_2 &= (0, s_{22}) = (0, \sqrt{E_b})
\end{aligned}
$$



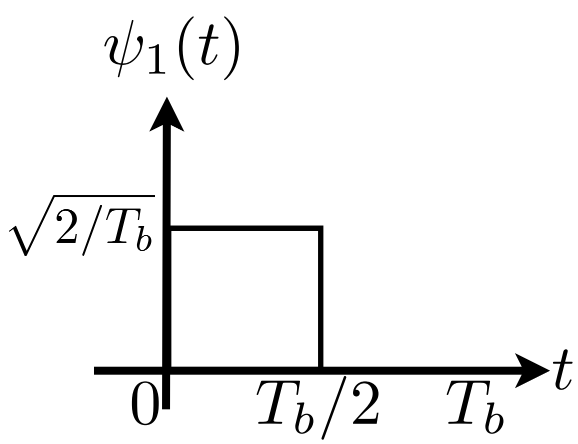
Example of two orthogonal signals

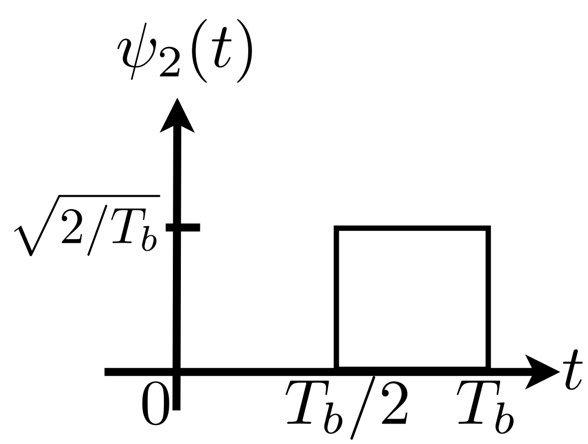




- Geometric representation

$$
\begin{aligned}
s_1(t) &= s_{11}\psi_1(t) + s_{12}\psi_2(t) \\
s_2(t) &= s_{12}\psi_1(t) + s_{22}\psi_2(t)
\end{aligned}
$$

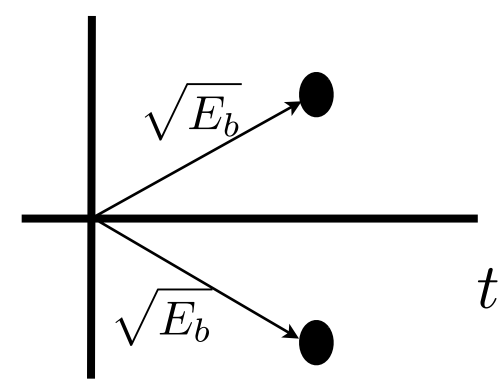




$$
\begin{aligned}
s_{11} &= \int_0^{T_b} s_1(t)\psi_1(t)\,dt = \sqrt{E_b/2} \\
s_{12} &= \int_0^{T_b} s_1(t)\psi_2(t)\,dt = \sqrt{E_b/2} \\
s_{21} &= \int_0^{T_b} s_2(t)\psi_1(t)\,dt = \sqrt{E_b/2} \\
s_{22} &= \int_0^{T_b} s_2(t)\psi_2(t)\,dt = -\sqrt{E_b/2}
\end{aligned}
$$

- Vector representation

$$
\begin{aligned}
\mathbf{s}_1 &= (\sqrt{E_b/2}, \sqrt{E_b/2}) \\
\mathbf{s}_2 &= (\sqrt{E_b/2}, -\sqrt{E_b/2})
\end{aligned}
$$



Gram-Schmidt Procedure

- Suppose that we have a set of finite energy signal waveforms $\{s_i(t),\ i = 1, 2, \ldots, M\}$ and we wish to construct a set of orthonormal waveforms $\{\psi_n(t)\}_{n=1}^{N}$.
  - The Gram-Schmidt procedure allows us to construct such a set!

- Gram-Schmidt procedure
  - Step 1: Begin with the first waveform $s_1(t)$, which is assumed to have energy E_1. The first orthonormal waveform is simply constructed as

$$\psi_1(t) = \frac{s_1(t)}{E_1}$$

- Step 2: The second waveform is constructed from $s_2(t)$ by first computing the projection of $\psi_1(t)$ onto $s_2(t)$, which is

$$c_{12} = \int_{-\infty}^{\infty} s_2(t)\psi_1(t)\,dt$$

- Then $c_{12}\psi_1(t)$ is subtracted from $s_2(t)$ to yield

$$d_2(t) = s_2(t) - c_{21}\psi_1(t)$$

- Now, $d_2(t)$ is orthogonal to $\psi_1(t)$, but it does not possess unit energy.

- If $\mathcal{E}_2$ denotes the energy in $d_2(t)$, then the energy-normalized waveform that is orthogonal to $\psi_1(t)$ is

$$\psi_2(t) = \frac{d_2(t)}{\sqrt{\mathcal{E}_2}}; \qquad \mathcal{E}_2 = \int_{-\infty}^{\infty} d_2^2(t)\,dt$$

- In general, the orthogonalization of the *k*-th function leads to

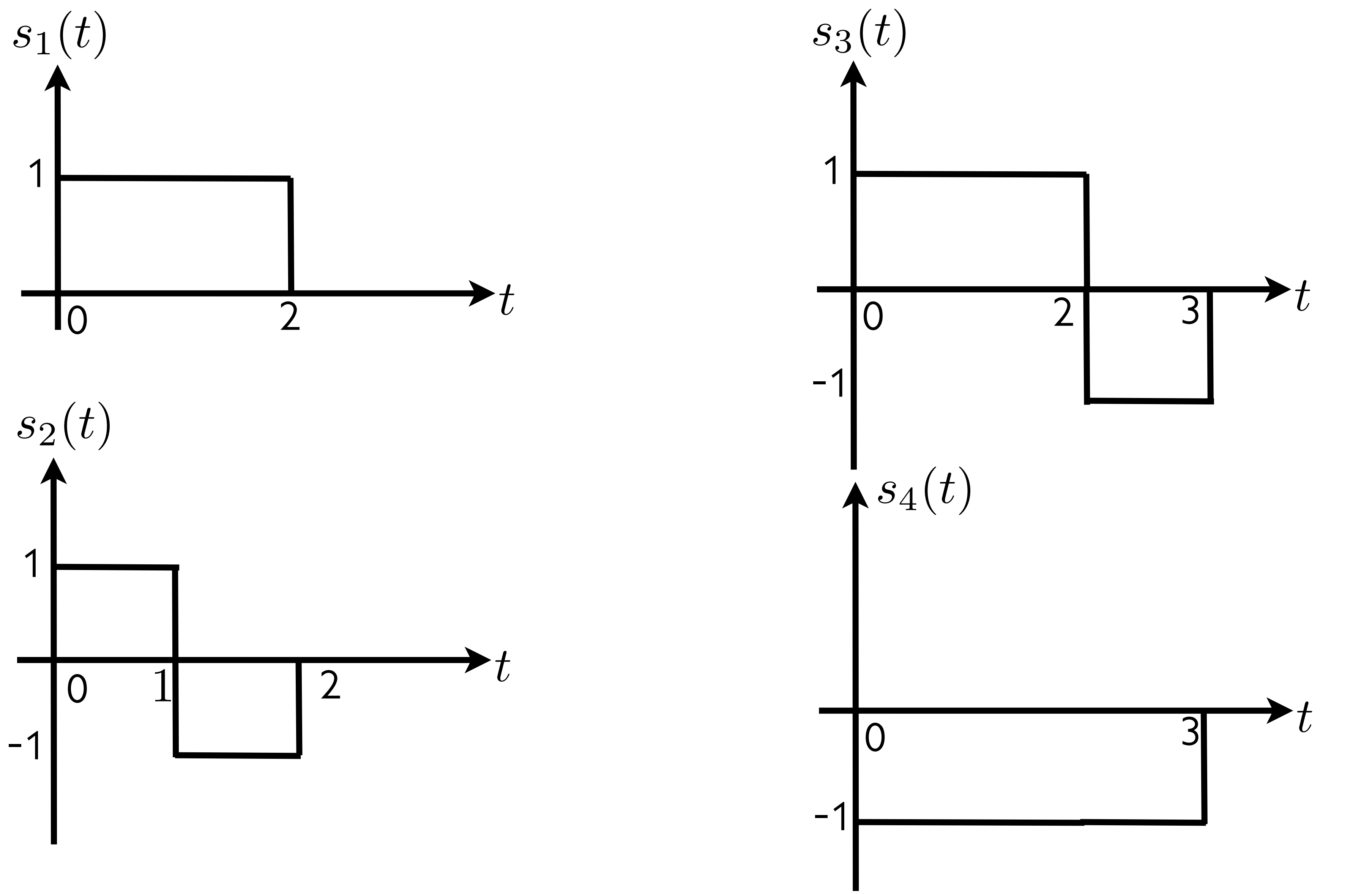
$$\psi_k(t) = \frac{d_k(t)}{\sqrt{\mathcal{E}_k}}$$

  - where

$$\begin{aligned} d_k(t) &= s_k(t) - \sum_{i=1}^{k-1} c_{ki}\psi_i(t) \\ c_{ki} &= \int_{-\infty}^{\infty} s_k(t)\psi_i(t)\,dt \\ \mathcal{E}_k &= \int_{-\infty}^{\infty} d_k^2(t)\,dt \end{aligned}$$

Example

- Find the orthonormal functions for the set of four waveforms $\{s_k(t)\}_{k=1}^{4}$



Gram-Schmidt procedure

- The waveform $s_1(t)$ has energy $\mathcal{E}_1 = 2$, so that

$$\psi_1(t) = \sqrt{\frac{1}{2}}s_1(t)$$

- We observe that $c_{12} = 0$. Hence, $s_2(t)$ are orthogonal to $\psi_1(t)$. Therefore,

$$\phi_2(t) = \frac{s_2(t)}{\sqrt{\mathcal{E}_2}}$$

- To obtain $\phi_3(t)$, we compute c_{13} and c_{23}, which are $c_{13} = \sqrt{2}$ and $c_{23} = 0$. Thus,

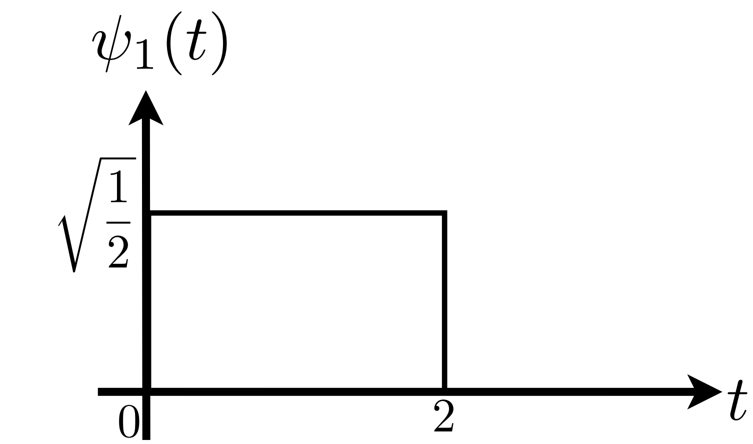
$$d_3(t) = s_3(t) - \sqrt{2}\psi_1(t) = \begin{cases} -1, & (2 \le t \le 3) \\ 0, & (\text{otherwise}). \end{cases}$$

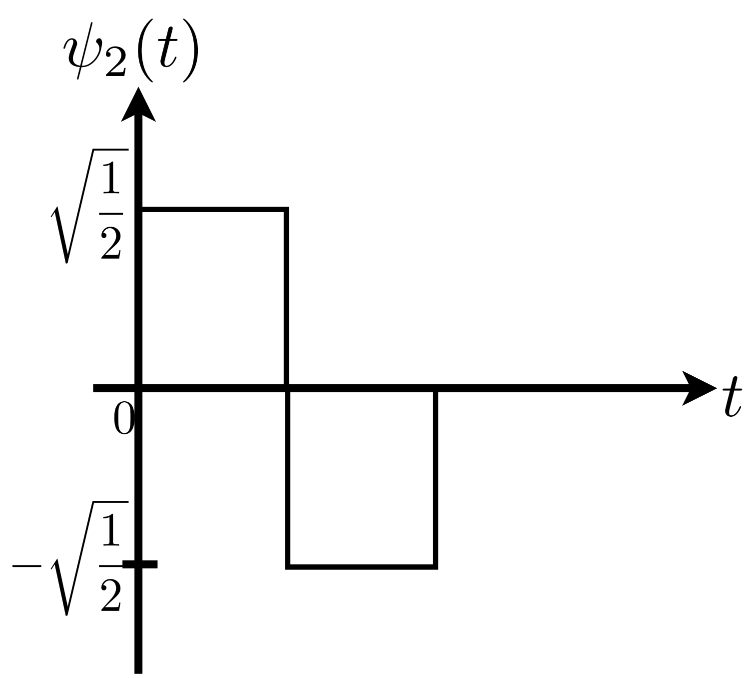
  - Since $d_3(t)$ has unit energy, it follows that $\psi_3(t) = d_3(t)$.

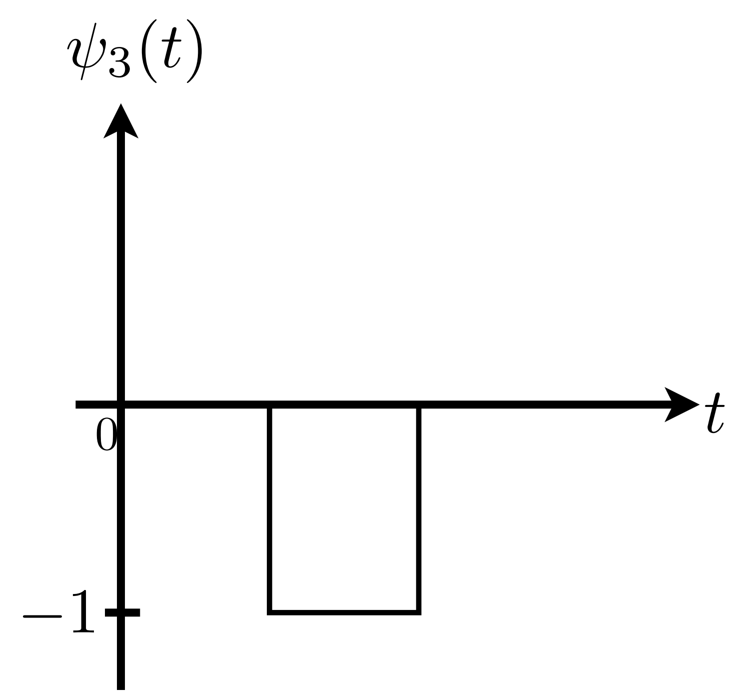
- In determining $\psi_4(t)$, we find that $c_{14} = -\sqrt{2}$, $c_{24} = 0$, and $c_{34} = 1$. Hence,

$$d_4(t) = s_4(t) + \sqrt{2}\phi_1(t) - \psi(t) = 0$$

- Consequently, $s_4(t)$ is a linear combination of $\psi_1(t)$ and $\psi_3(t)$, hence, $\psi_4(t) = 0$.







Geometrical Representation of Signals

- Once we have constructed the set of orthogonal waveforms $\{\psi_n(t)\}_{n=1}^{N}$, we can express the signals $\{s_m(t)\}_{m=1}^{M}$ as exact combinations of the $\{\psi_n(t)\}_{n=1}^{N}$.
  - Hence, we may write

Dimension

$$s_m(t) = \sum_{n=1}^{N} s_{mn}\psi_n(t), \quad m = 1, 2, \ldots, M.$$

where $$s_{mn} = \int_{-\infty}^{\infty} s_m(t)\psi_n(t)\, dt.$$

- Signal energy

$$\mathcal{E}_m = \int_{-\infty}^{\infty} s_m^2(t)\, dt = \sum_{n=1}^{N} s_{mn}^2.$$

- Vector representation

  - For $s_m(t) = \sum_{n=1}^{N} s_{mn}\phi_n(t)$, the vector representation of $s_m(t)$ is defined as

$$\mathbf{s}_m = \begin{bmatrix} s_{m1} & s_{m2} & \cdots & s_{mN} \end{bmatrix}$$

- Inner product of two signals

$$\mathbf{s}_m \cdot \mathbf{s}_n = \int_{-\infty}^{\infty} s_m(t) s_n(t)\, dt \quad = \sum_{k=1}^{N} s_{mk} s_{nk}$$

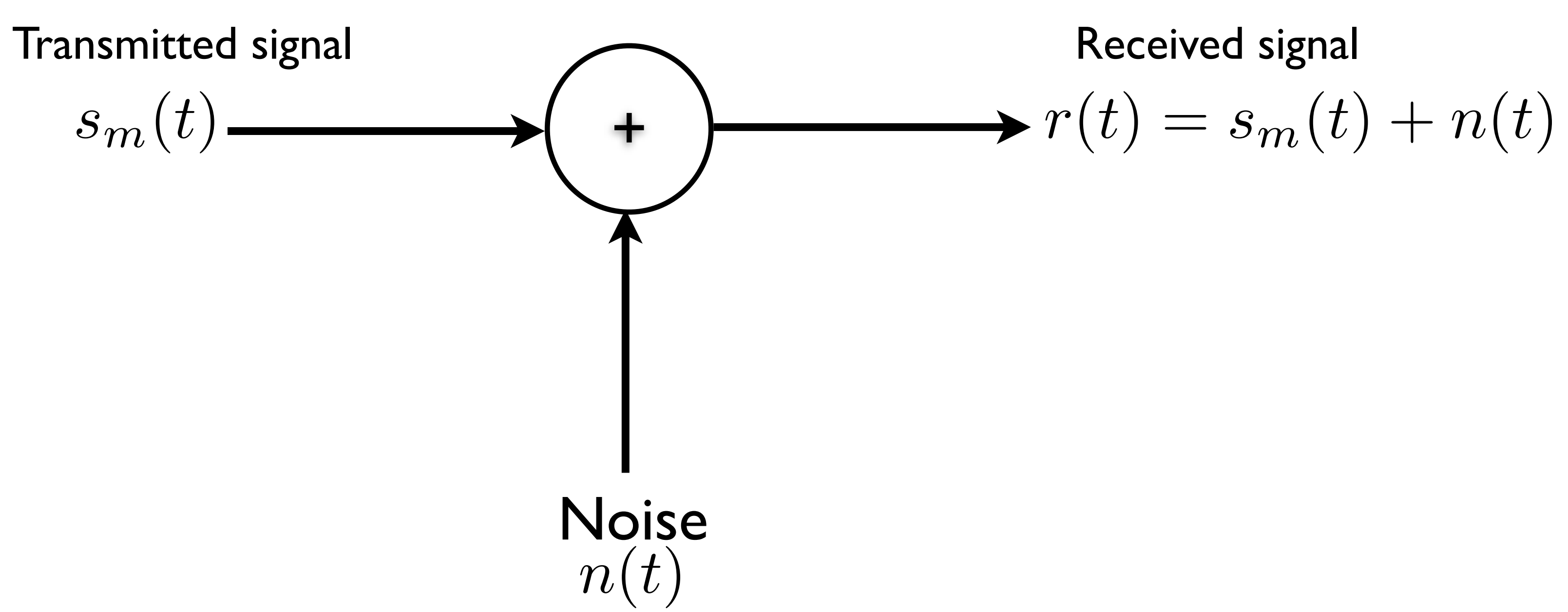
Additive White Gaussian Noise Channel

- Received signal in a signal interval of duration T_b over AWGN channel

$$r(t) = s_m(t) + n(t), \quad m = 1, 2,$$

  - $n(t)$ denotes the sample function of the additive white Gaussian noise (AWGN) process with the power spectral density $S_n(f) = N_0/2 \,\mathrm{W/Hz}.$

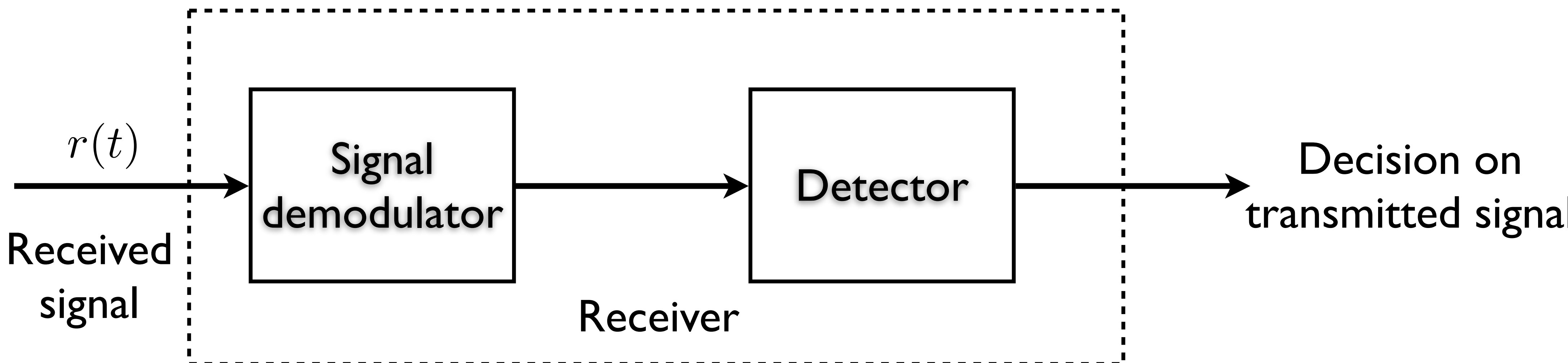
- Block diagram of AWGN channel



Optimum Receiver over AWGN

- Based on the observation of $r(t)$ over the signal interval, we wish to design a receiver that is optimum *in the sense that it minimizes the probability of making an error.*

- Receiver structure



- Two types of signal demodulator
  - Correlation-type demodulator
  - Matched filter-type demodulator

Correlation-Type Demodulator for Binary Antipodal Signals

- Signal waveform

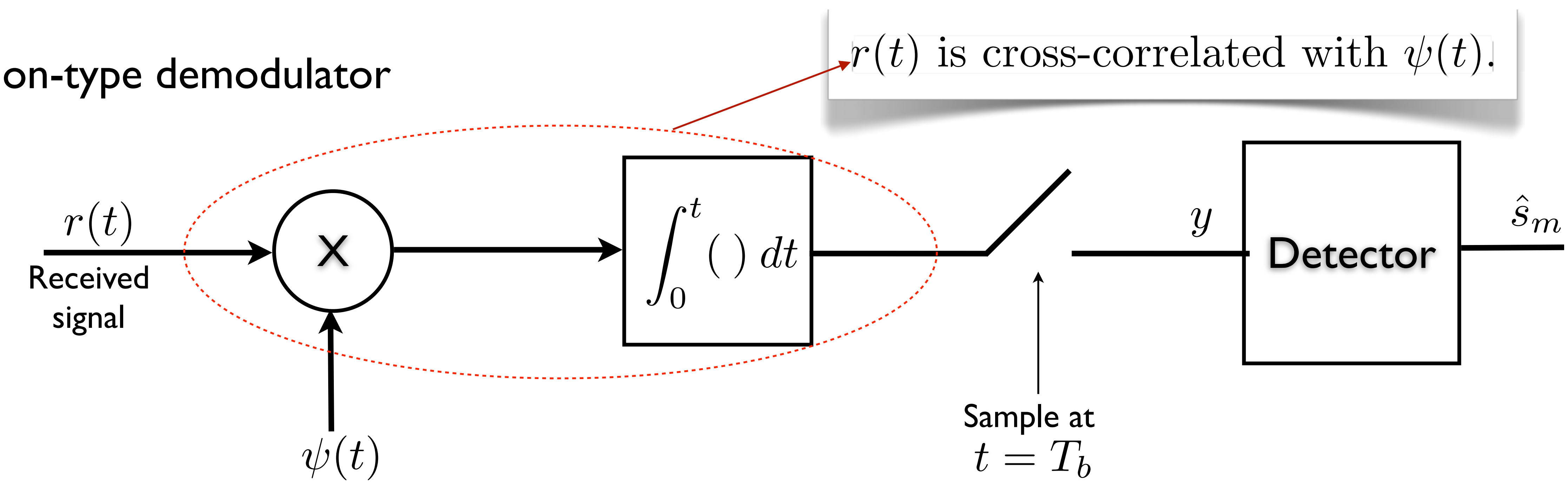
$$s_m(t) = s_m \psi(t), \;\; m = 1, 2$$

  - where $\psi(t)$ is the unit energy rectangular pulse and $s_1 = \sqrt{\mathcal{E}_b}, \;\; s_2 = -\sqrt{\mathcal{E}_b}$.

- Received signal

$$r(t) = s_m \psi(t) + n(t), \;\; 0 \le t \le T_b, \;\; m = 1, 2.$$

- Correlation-type demodulator



- Output of cross-correlation operation

$$
\begin{aligned}
y(t) &= \int_0^t r(\tau)\psi(\tau)\,d\tau \\
&= \int_0^t [s_m\psi(\tau) + n(\tau)]\psi(\tau)\,d\tau \\
&= s_m \int_0^t \psi^2(\tau)\,d\tau + \int_0^t n(t)\psi(\tau)\,d\tau.
\end{aligned}
$$

- Sampling the output of the correlator at $t = T_b$

$$y(T_b) = s_m + n$$

desired signal term noise term

  - where

$$n = \int_0^{T_b} \psi(\tau)n(\tau)\,d\tau$$

- Noise term

$$n = \int_0^{T_b} \psi(\tau) n(\tau)\, d\tau$$

  - n is Gaussian random variable.
  - Mean

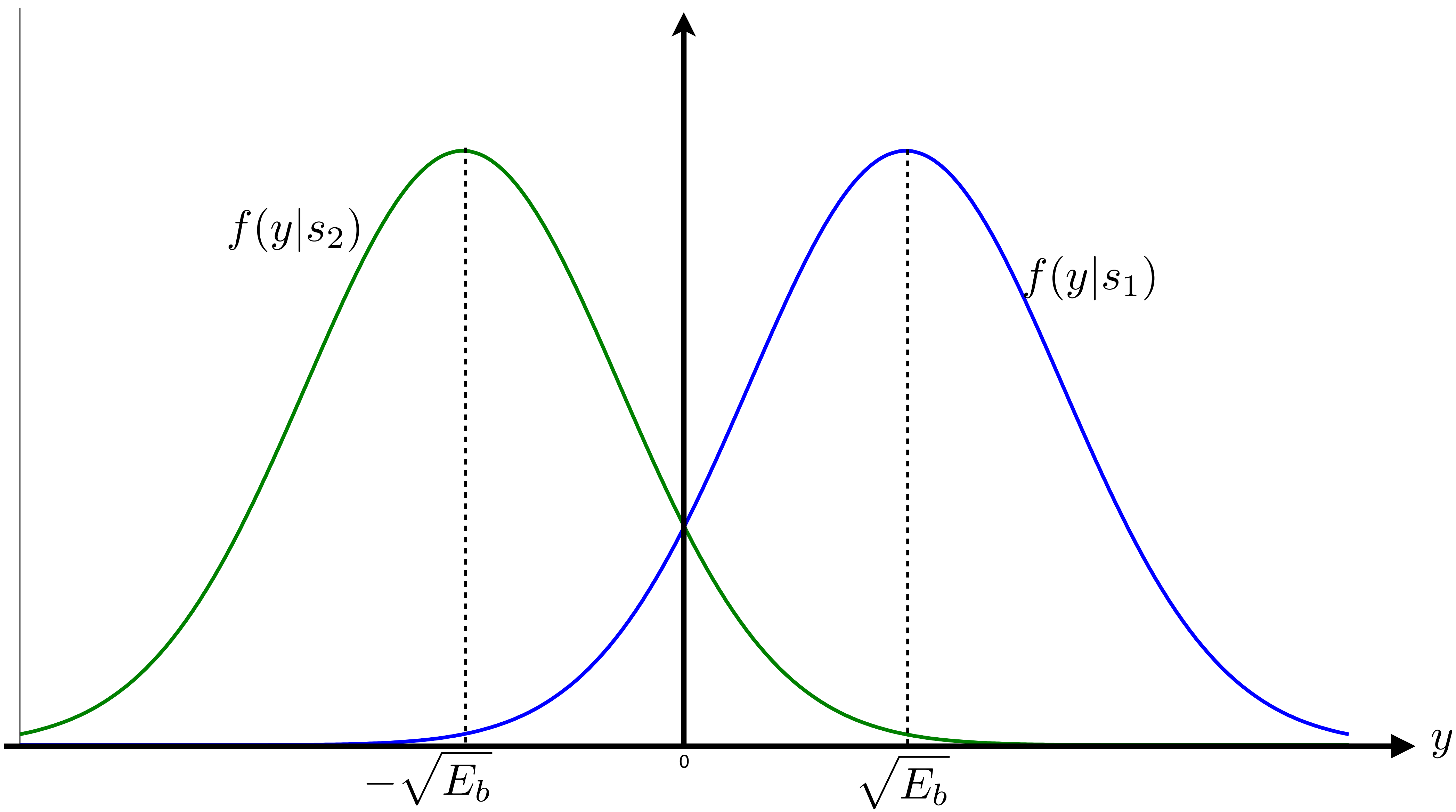
$$E[n] = E\left[\int_0^{T_b} \psi(\tau) n(\tau)\, d\tau\right] = \int_0^{T_b} \psi(\tau) E[n(\tau)]\, d\tau = 0$$

  - Variance

$$\begin{aligned} \sigma_n^2 &= E[n^2] = \int_0^{T_b}\int_0^{T_b} E[n(t)n(\tau)]\psi(t)\psi(\tau)\, dt d\tau \\ &= \int_0^{T_b}\int_0^{T_b} \frac{N_0}{2}\delta(t-\tau)\psi(t)\psi(\tau)\, dt d\tau \\ &= \frac{N_0}{2}\int_0^{T_b} \psi^2(t)\, dt = \frac{N_0}{2}. \end{aligned}$$

- Conditional PDF given s_m

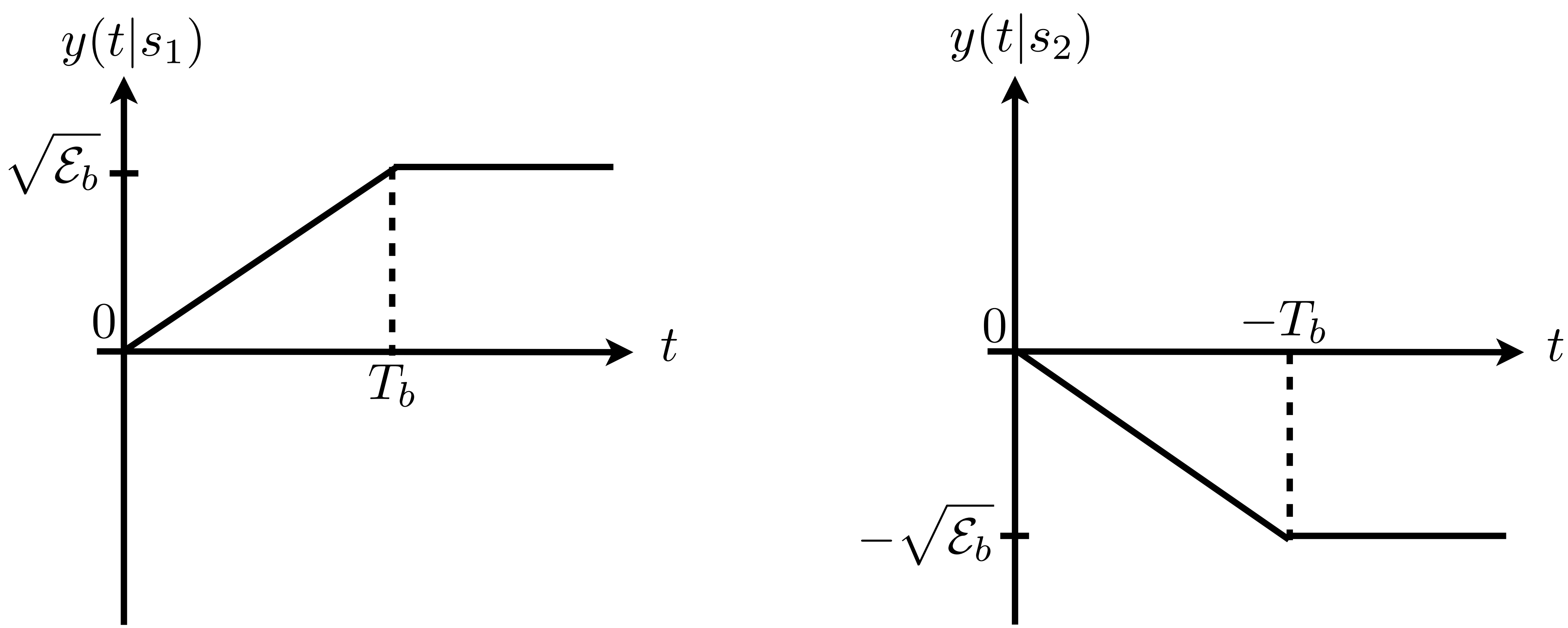
$$f(y|s_m) = \frac{1}{\sqrt{\pi N_0}} e^{-(y-s_m)^2/N_0}, \quad m = 1, 2.$$



- Noise-free output of the correlator for the rectangular pulse $\psi(t)$

With $n(t) = 0$, the signal waveform at the output of the correlator is

$$y(t) = \int_0^t s_m \psi^2(\tau)\, d\tau = s_m \int_0^t \psi^2(t)\, d\tau$$



- Note that the maximum signal at the output of the correlator occurs at $t = T_b$.

- We also observe that the correlator must be reset to zero at the end of each bit interval T_b, so that it can be used in the demodulator of the received signal in the next signal interval. Such an integrator is called an integrate-and-dump filer.

Correlation-Type Demodulator for Binary Orthogonal Signals

- Signal waveform

$$r(t) = s_m(t) + n(t), \quad 0 \le t \le T_b, \quad m = 1, 2.$$

where $s_1(t) = \sqrt{\mathcal{E}_b}\psi_1(t)$, and $s_2(t) = \sqrt{\mathcal{E}_b}\psi_2(t)$

  - Note that in vector form, the transmit signals are

$$\mathbf{s}_1 = [\sqrt{\mathcal{E}_b},\ 0], \text{ and } \mathbf{s}_2 = [0,\ \sqrt{\mathcal{E}_b}]$$

- Correlation-type demodulator