Next we compute the wavefunction

$$a|0\rangle = 0 \qquad a = \frac{1}{\sqrt{2}}\left(\xi + \frac{d}{d\xi}\right), \quad \xi = \sqrt{\frac{m\omega}{\hbar}}\,x$$

$$\int d^3x\, |x\rangle \left(\xi + \frac{d}{d\xi}\right) \langle x|0\rangle = 0$$

$$\frac{d}{d\xi}\psi_0(x) = -\xi\psi_0(x)$$

$$\frac{d\psi_0}{\psi_0} = -\xi\, d\xi$$

$$\log\frac{\psi_0(x)}{\psi_0(0)} = -\frac{1}{2}\xi^2$$

$$\psi_0(x) = Ce^{-\frac{m\omega}{2\hbar}x^2} \qquad C = \psi_0(0)$$

$$\int_{-\infty}^{\infty} |\psi_0(x)|^2 dx = C^2 \int_{-\infty}^{\infty} dx\; e^{-\frac{m\omega}{\hbar}x^2}$$

$$= C^2\sqrt{\frac{\hbar}{m\omega}} \int_{-\infty}^{\infty} dt\; e^{-t^2}$$

$$= C^2\sqrt{\frac{\hbar}{m\omega}}\sqrt{\pi}$$

$$\therefore\; C = \sqrt[4]{\frac{m\omega}{\hbar\pi}}$$

$$\Rightarrow \quad \psi_0(x) = \sqrt[4]{\frac{m\omega}{\hbar\pi}}\, e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\omega \to \omega' = \omega\sqrt{1+\varepsilon}$$

$$\psi_0(x)| = \sqrt[4]{\frac{m\omega'}{\hbar\pi}}\, e^{-\frac{m\omega'}{2\hbar}x^2}$$

We can make an expansion of perturbed $\psi_0(x)$ in powers of ε

$$\psi_0(x) = \sqrt[4]{\frac{m\omega}{\hbar\omega}}(1+\varepsilon)^{\frac{1}{8}}\, e^{-\frac{m\omega}{2\hbar}x^2 \times\sqrt{1+\varepsilon}}$$

$$e^{-a\sqrt{1+\varepsilon}} = e^{-a}\left[1+\varepsilon\frac{d}{d\varepsilon}\left(-a\sqrt{1+\varepsilon}\right)+\frac{\varepsilon^2}{2}\frac{d^2}{d\varepsilon^2}\left(-a\sqrt{1+\varepsilon}\right)+\cdots\right]$$

$$= e^{-a}\left[1+\varepsilon\left[(-a)\frac{1}{2\sqrt{1+\varepsilon}}\right]_{\varepsilon=0}+O(\varepsilon^2)\right]$$

$$= e^{-a}\left[1-\frac{a}{2}\varepsilon+O(\varepsilon^2)\right]$$

$$\Rightarrow e^{-\frac{m\omega}{2\hbar}x^2}\left[1-\frac{m\omega}{4\hbar}x^2\varepsilon+O(\varepsilon^2)\right]$$

$$(1+\varepsilon)^{\frac{1}{8}} = \left(1+\frac{1}{8}\varepsilon+O(\varepsilon^2)\right)$$

$$\therefore \psi_0(x) = \sqrt[4]{\frac{m\omega}{\hbar\pi}}\, e^{-\frac{m\omega}{2\hbar}x^2}\left[1+\varepsilon\left(\frac{1}{8}-\frac{m\omega}{4\hbar}x^2\right)+O(\varepsilon^2)\right]$$

$$\langle x|0\rangle = \sqrt[4]{\frac{m\omega}{\hbar\pi}}\, e^{-\frac{m\omega}{2\hbar}x^2} \qquad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$$

$$a^\dagger = \frac{1}{\sqrt{2}}\left(\xi-\frac{d}{d\xi}\right)$$

$$|1\rangle = a^\dagger|0\rangle$$

$$\langle x|1\rangle = \frac{1}{\sqrt{2}}\left(\xi-\frac{d}{d\xi}\right)\langle x|0\rangle$$

$$\langle x|2\rangle = \left[\frac{1}{\sqrt{2}}\left(\xi-\frac{d}{d\xi}\right)\right]^2\langle x|0\rangle \qquad \xi = \sqrt{\frac{m\omega}{\hbar}}x$$

$$\ldots \sqrt{\frac{\ldots\omega}{\hbar\pi}}\, e^{\ldots 2} \qquad \xi = \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$\langle x|1\rangle = \frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right) \sqrt[4]{\frac{m\omega}{\hbar\pi}}\, e^{-\frac{\xi^2}{2}}$$

$$\left(\xi - \frac{d}{d\xi}\right) e^{-\frac{\xi^2}{2}} = e^{-\frac{\xi^2}{2}}\left[\xi - \frac{d}{d\xi}\left(-\frac{\xi^2}{2}\right)\right]$$

$$= 2\xi\, e^{-\frac{\xi^2}{2}}$$

$$= \sqrt[4]{\frac{m\omega}{\hbar\pi}}\,\frac{1}{\sqrt{2}}\, 2\sqrt{\frac{m\omega}{\hbar}}\, x\, e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= \left(\frac{m\omega}{\hbar}\right)^{\frac{3}{4}} \frac{\sqrt{2}}{\sqrt[4]{\pi}}\, x\, e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\langle x|2\rangle = \frac{1}{\sqrt{2}}\left[\left(\frac{1}{\sqrt{2}}\right)\left(\xi - \frac{d}{d\xi}\right)\right]^2 \sqrt[4]{\frac{m\omega}{\hbar\pi}}\, e^{-\frac{\xi^2}{2}}$$

$$= \frac{1}{2\sqrt{2}} \sqrt[4]{\frac{m\omega}{\hbar\pi}} \times \left(\xi - \frac{d}{d\xi}\right)\left(\xi - \frac{d}{d\xi}\right) e^{-\frac{\xi^2}{2}}$$

$$= \sqrt[4]{\frac{m\omega}{\hbar\pi}}\,\frac{1}{2\sqrt{2}}\left(\xi^2 - 1 - 2\xi\frac{d}{d\xi} + \frac{d^2}{d\xi^2}\right) e^{-\frac{\xi^2}{2}}$$

$$\frac{d}{d\xi} e^{-\frac{\xi^2}{2}} = \frac{d}{d\xi}\left(-\frac{\xi^2}{2}\right) = -\xi \quad \Rightarrow \quad -2\xi\frac{d}{d\xi} e^{-\frac{\xi^2}{2}} = 2\xi^2$$

$$\frac{d^2}{d\xi^2} e^{-\frac{\xi^2}{2}} = \frac{d}{d\xi}\left(-\xi\, e^{-\frac{\xi^2}{2}}\right) = (-1 + \xi^2)$$

$$= \sqrt[4]{\frac{m\omega}{\hbar\pi}}\,\frac{1}{2\sqrt{2}}\left[\xi^2 - 1 + 2\xi^2 - 1 + \xi^2\right] e^{-\frac{\xi^2}{2}}$$

$$= \sqrt[4]{\frac{m\omega}{\hbar\pi}}\,\frac{1}{2\sqrt{2}}\left[4\xi^2 - 2\right] e^{-\frac{\xi^2}{2}}$$

$$= \sqrt[4]{\frac{m\omega}{\hbar\pi}}\,\frac{1}{2\sqrt{2}}\left[\frac{4m\omega}{\hbar}x^2 - 2\right] e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\langle x|2\rangle = \sqrt[4]{\frac{m\omega}{\hbar\pi}}\,\frac{1}{2\sqrt{2}}\left[\frac{2m\omega}{\hbar}x^2-2\right]e^{-\frac{m\omega}{2\hbar}x^2} \qquad \frac{16}{2\sqrt{2}}$$

$$= \sqrt[4]{\frac{m\omega}{\hbar\pi}}\,4\sqrt{2}\left[\frac{m\omega}{4\hbar}x^2-\frac{1}{8}\right]e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\langle x|0\rangle = \sqrt[4]{\frac{m\omega}{\hbar\pi}}\,e^{-\frac{m\omega}{2\hbar}x^2}$$

We recall that

$$\psi_0(x)\Big|_{\text{perturbed}} = \sqrt[4]{\frac{m\omega}{\hbar\pi}}\,e^{-\frac{m\omega}{2\hbar}x^2}\left[1+\varepsilon\left(\frac{1}{8}-\frac{m\omega}{4\hbar}x^2\right)+O(\varepsilon^2)\right]$$

$$= \langle x|0\rangle - \frac{\varepsilon}{4\sqrt{2}}\langle x|2\rangle + O(\varepsilon^2).$$

Next, we compute the wavefunction using the perturbation theory.

$$|n\rangle = |n^{(0)}\rangle + \sum_{k\neq n}|k^{(0)}\rangle\frac{V_{kn}}{E_n^{(0)}-E_k^{(0)}}.$$

$$|0\rangle = |0^{(0)}\rangle + |2^{(0)}\rangle\frac{\langle 2^{(0)}|V|0^{(0)}\rangle}{E_0^{(0)}-E_2^{(0)}} \qquad V=\frac{1}{2}\varepsilon m\omega^2x^2$$

$$\left.\begin{aligned} V_{20} &= \frac{\varepsilon}{2\sqrt{2}}\hbar\omega \\ E_{(0)}^{(0)}-E_2^{(0)} &= -2\hbar\omega \end{aligned}\right\}$$

$$= |0^{(0)}\rangle - |2^{(0)}\rangle\frac{\varepsilon}{4\sqrt{2}} + O(\varepsilon^2)$$

The perturbation theory reproduces the result !.

QM 2
Graduate 3

Degenerate Perturbation Theory

$\{|m^{(0)}\rangle\}$ is a set of linearly independent eigenkets of g. $\dim\{|m^{(0)}\rangle\} = g$. They are all degenerate

$$H_0|m^{(0)}\rangle = E_D^{(0)}|m^{(0)}\rangle \quad , \quad m = 1, 2, \cdots, g.$$

Under perturbation λV, $(H = H_0 + \lambda V$ $D = \{1, 2, \cdots, g\}$.

$$(H_0 + \lambda V)|l\rangle = (E_D^{(0)} + \Delta_l)|l\rangle.$$

Note that $|l^{(0)}\rangle$ must be within the space spanned by $\{|m^{(0)}\rangle\}$. However, it does not have to be identical to a single state $|m^{(0)}\rangle$. Instead, it ~~may b~~ is, in general, a linear combination: $|l^{(0)}\rangle = \sum_{m\in D} |m^{(0)}\rangle\langle m^{(0)}|l^{(0)}\rangle$, where $\lim_{\lambda\to 0}|l\rangle = |l^{(0)}\rangle$.

$$(E_D^{(0)} - H_0)|l\rangle = (\lambda V - \Delta_l)|l\rangle$$

$|l\rangle$ can be expanded as

$$|l\rangle = |l^{(0)}\rangle + \lambda|l^{(1)}\rangle + \lambda^2|l^{(2)}\rangle + \cdots$$

$$(E_D^{(0)} - H_0)(|l^{(0)}\rangle + \lambda|l^{(1)}\rangle + \cdots)$$
$$= [\lambda(V - \Delta_l^{(1)}) - \lambda^2\Delta_l^{(2)} \cdots](|l^{(0)}\rangle + \lambda|l^{(1)}\rangle + \cdots)$$

order $\lambda^{(0)}$: $\quad (E_D^{(0)} - H_0)|l^{(0)}\rangle = 0$

order $\lambda^{(1)}$: $\quad (E_D^{(0)} - H_0)|l^{(1)}\rangle = (V - \Delta_l^{(1)})|l^{(0)}\rangle$

multiplying $\langle m^{(0)}|$ to the left, we find that

$$\langle m^{(0)}|V|l^{(0)}\rangle = \Delta_l^{(1)}\langle m^{(0)}|l^{(0)}\rangle$$

$\langle m^{(0)}| V \, \mathbb{1} |l^{(0)}\rangle = \Delta_l^{(1)} \langle m^{(0)}|l^{(0)}\rangle$

$\mathbb{1} = \sum_{m'\in D} |m'^{(0)}\rangle\langle m'^{(0)}|$

$\langle m^{(0)}|V|m'^{(0)}\rangle\langle m'^{(0)}|l^{(0)}\rangle = \Delta_l^{(1)}\langle m^{(0)}|l^{(0)}\rangle$

$V_{mm'} l_{m'} = \Delta_l^{(1)} l_m$

$\Rightarrow VL = \Delta_l^{(1)} L \qquad L = \begin{pmatrix} \langle 1^{(0)}|l^{(0)}\rangle \\ \vdots \\ \langle g^{(0)}|l^{(0)}\rangle \end{pmatrix}$

is a colum vector

It is an eigenvalue equation!

$(V - \Delta_l^{(1)}\mathbb{1}) L = 0.$

In order to have a non-trivial ($L \neq 0$) solution, we must have

$$\det(V - \Delta_l^{(1)}\mathbb{1}) = 0.$$

We return to order-λ equation

$$(E_D^{(0)} - H_0)|l^{(1)}\rangle = (V - \Delta_l^{(1)})|l^{(0)}\rangle$$

We define the projection operator

$$\phi_D = 1 - \sum_{m\in D}|m^{(0)}\rangle\langle m^{(0)}| = \sum_{k\notin D}|k^{(0)}\rangle\langle k^{(0)}|$$

that removes the components in the vector space generated by $\{|m^{(0)}\rangle\}$.

$$\phi_D(E_D^{(0)}-H_0)|l^{(1)}\rangle = \phi_D(V-\Delta_l^{(1)})|l^{(0)}\rangle$$

Because $\langle m^{(0)}|(E_D^{(0)}-H_0)|l^{(1)}\rangle = 0$, $\left(\langle m^{(0)}|H_0=\langle m^{(0)}|E_D^{(0)}\right)$

$(V-\Delta_l^{(1)})|l^{(0)}\rangle$ does not have any component $|m^{(0)}\rangle$, $m\in D$.

Therefore,

$$(V-\Delta_l^{(1)})|l^{(0)}\rangle = \phi_D(V-\Delta_l^{(1)})|l^{(0)}\rangle.$$

In addition, because $|l^{(0)}\rangle$ is a linear combination of $\{|m^{(0)}\rangle\}$,

$$\phi_D|l^{(0)}\rangle = 0 \rightarrow \phi_D\Delta_l^{(1)}|l^{(0)}\rangle = 0.$$

Therefore, $\qquad = \phi_D V|l^{(0)}\rangle.$

$$\Rightarrow \quad \phi_D(E_D^{(0)}-H_0)|l^{(1)}\rangle = \phi_D V|l^{(0)}\rangle$$

$$\Rightarrow \quad |l^{(1)}\rangle = \phi_D\frac{1}{E_D^{(0)}-H_0}\phi_D V|l^{(0)}\rangle$$

$$\boxed{|l^{(1)}\rangle = \phi_D\frac{1}{E_D^{(0)}-H_0}V|l^{(0)}\rangle}$$

$$\boxed{=\sum_{k\notin D}\frac{|k^{(0)}\rangle}{E_D^{(0)}-E_k^{(0)}}\langle k^{(0)}|V|l^{(0)}\rangle}$$

Order-λ^2 equation is

$$(E_D^{(0)}-H_0)|l^{(2)}\rangle = -\Delta_l^{(2)}|l^{(0)}\rangle - (V-\Delta_l^{(1)})|l^{(1)}\rangle$$

$$\Rightarrow \langle l^{(0)}|(E_D^{(0)}-H_0)|l^{(2)}\rangle = -\Delta_l^{(2)}$$

$\qquad \overset{\shortparallel}{0}$

$$\boxed{\begin{aligned}\Delta_l^{(2)} &= \langle l^{(0)}|(V-\Delta_l^{(1)})|l^{(1)}\rangle \\ &= \langle l^{(0)}|V\frac{\phi_D}{E_D^{(0)}-H_0}V|l^{(0)}\rangle\end{aligned}}$$

$$\Delta_\ell^{(2)} = \sum_{k \notin D} \frac{\langle \ell^{(0)} | V | k^{(0)} \rangle \langle k^{(0)} | V | \ell^{(0)} \rangle}{E_D^{(0)} - E_k^{(0)}} = \sum_{k \notin D} \frac{|V_{k\ell}|^2}{E_D^{(0)} - E_k^{(0)}}$$

We return to the all-order expression:

$$(E_D^{(0)} - H_0)|\ell\rangle = (\lambda V - \Delta_\ell)|\ell\rangle$$

Because $\langle m^{(0)}|(E_D^{(0)} - H_0)|\ell\rangle = 0$

to any order in λ,

$$\langle m^{(0)}|(\lambda V - \Delta_\ell)|\ell\rangle = 0$$

$$\Rightarrow \Delta_\ell \langle m^{(0)}|\ell\rangle = \lambda \langle m^{(0)}|V|\ell\rangle$$

to all orders in λ.

As in the non-~~perturbative~~ degenerate case, it is convenient to choose the normalization

$$\langle \ell^{(0)}|\ell\rangle = 1$$

$$\therefore \quad \Delta_\ell = \lambda \langle \ell^{(0)}|V|\ell\rangle$$

Linear stark effect.

Hydrogen atom states

$|n l m\rangle$

If $n=1$, then $l=0$ and $m=0$.

$\therefore$ $|100\rangle$ is the only state whose energy is $E_n = E_1 = -\frac{1}{2}\mu c^2 (Z\alpha)^2$.

If $n=2$, then $l = \begin{cases} 1 \\ 0 \end{cases}$ $m = -1, 0, 1$.

$\left(\therefore \begin{matrix} |200\rangle \\ |21{-1}\rangle, |210\rangle, |211\rangle \end{matrix} \right)$

There are 4 states with the energy eigenvalue

$$E_n = E_2 = -\frac{1}{2}\mu c^2 \frac{(Z\alpha)^2}{2^2}$$

Note that

$$E_n = -\frac{1}{2}\mu c^2 \frac{(Z\alpha)^2}{n^2} \text{ and } \alpha = \frac{e^2 k_1}{\hbar c} \simeq \frac{1}{137}$$

where $k_1 = \begin{cases} \frac{1}{4\pi\varepsilon_0} & \text{MKSA unit system} \\ \frac{1}{4\pi} & \text{Heaviside-Lorentz unit} \\ 1 & \text{Gaussian unit.} \end{cases}$

$$\therefore \alpha = \begin{cases} \frac{e^2}{4\pi\varepsilon_0\hbar c} & \text{MKSA} \\ \frac{e^2}{4\pi\hbar c} & \text{HL} \\ e^2/(\hbar c) & \text{G.} \end{cases}$$

We introduce the Bohr radius

$$a_0 = \frac{\hbar}{\mu c \alpha} = \begin{cases} \dfrac{4\pi\varepsilon_0 \hbar^2}{\mu e^2} & \text{MKSA} \\ 4\pi\hbar^2/(\mu e^2) & \text{HL} \\ \dfrac{\hbar^2}{\mu e^2} & \text{G.} \end{cases}$$

Note that

$$E_n = -\frac{1}{2}\mu c^2 \frac{(Z\alpha)^2}{n^2} = -\frac{1}{2}(\mu c \alpha)\frac{(cZ^2\alpha)}{n^2}$$

$$\mu c \alpha = \frac{\hbar}{a_0} \qquad = -\frac{1}{2}\frac{\hbar}{a_0}\frac{cZ^2\alpha}{n^2}.$$

$$\alpha = \frac{k_1 e^2}{\hbar c} \qquad = -\frac{1}{2}\frac{\cancel{\hbar c}}{a_0}\cdot\frac{Z^2}{n^2}\cdot\frac{k_1 e^2}{\cancel{\hbar c}}$$

$$= -\frac{1}{2}\left(\frac{k_1 e^2}{a_0}\right)\frac{Z^2}{n^2}$$

$$\therefore E_n = \begin{cases} -\dfrac{1}{2}\left(\dfrac{e^2}{4\pi\varepsilon_0 a_0}\right)\dfrac{Z^2}{n^2} & \text{MKSA} \\ -\dfrac{1}{2}\left(\dfrac{e^2}{4\pi a_0}\right)\dfrac{Z^2}{n^2} & \text{HL} \\ -\dfrac{1}{2}\left(\dfrac{e^2}{a_0}\right)\dfrac{Z^2}{n^2} & \text{G} \end{cases}$$