Communication Signals

(Haykin Sec. 2.1 - Sec. 2.2 and Ziemer Sec. 2.5)

KECE321 Communication Systems I

Lecture #5, March 19, 2012

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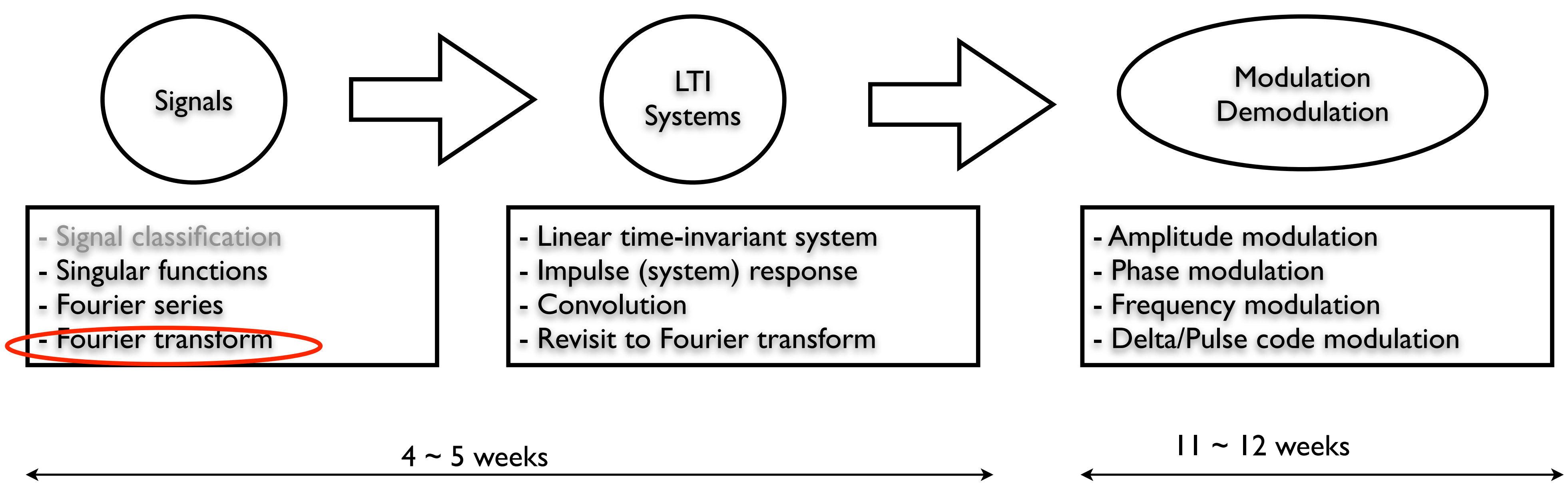
Review

- Generalized Fourier series
    - Integral-square error
- Complex exponential Fourier series

Summary of Today's Lecture

- Fourier transform
  - Definition
  - Continuous spectrum
  - Properties



Fourier Transform

- Now we want to generalize the Fourier series to represent aperiodic signals using the Fourier series form given as

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}, \qquad t_0 \leq t \leq t + 0 + T_0$$

$$X_n = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jn\omega_0 t}\, dt$$

- Consider non-periodic signal $x(t)$ but is an energy signal.
  - In the interval $|t| < \frac{1}{2}T_0$, we can represent $x(t)$ as

$$x(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(\lambda) e^{-j2\pi n f_0 \lambda}\, d\lambda \right] e^{jn2\pi n f_0 t}, \qquad |t| < \frac{T_0}{2}$$

    - where $f_0 = 1/T_0$.
  - To represent $x(t)$ for all time, we simply let $T_0 \to \infty$ such that

$$nf_0 = n/T_0 \to f, \quad 1/T_0 \to df, \quad \sum_{n=-\infty}^{\infty} \to \int_{-\infty}^{\infty}$$

- Thus

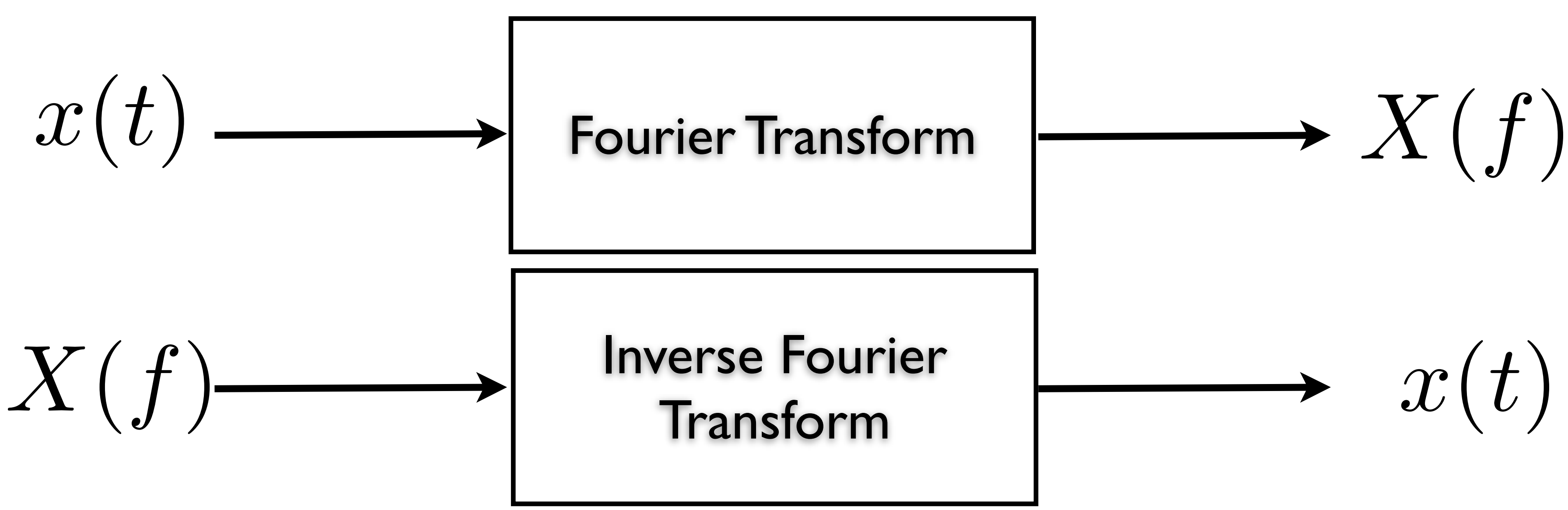
$$x(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f \lambda} \, d\lambda \right] e^{j2\pi f t} \, df$$

- Defining

$$X(f) = \int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f \lambda} \, d\lambda$$

we can rewrite

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$



- Notations

$$X(f) = \mathcal{F}[x(t)]$$

$$x(t) = \mathcal{F}^{-1}[X(f)]$$

$$x(t) \Longleftrightarrow X(f)$$

Amplitude and Phase Spectra

- Writing $X(f)$ in phasor form:

$$X(f) = |X(f)|e^{j\theta(f)}, \qquad \theta(f) = \angle X(f)$$

- we can show that for real $x(t)$, that

$$|X(f)| = |X(-f)| \text{ and } \theta(-f) = -\theta(f)$$

- This is done by Euler's theorem to write

$$\begin{aligned} R &= \Re X(f) = \int_{-\infty}^{\infty} x(t)\cos(2\pi f t)\,dt \\ I &= \Im X(f) = -\int_{-\infty}^{\infty} x(t)\sin(2\pi f t)\,dt \end{aligned}$$

- Then, the square of amplitude and the phase are

$$|X(f)|^2 = R^2 + I^2, \qquad \theta(f) = \tan^{-1}\left(\frac{I}{R}\right)$$

- Amplitude spectrum: Plot of $|X(f)|$ versus f
- Phase spectrum: Plot of $\angle X(f)$ versus f

Example

- Fourier transform of rectangular pulse $g(t) = A\,\mathrm{rect}\left(\frac{t}{T}\right)$

$$
\begin{aligned}
G(f) &= \int_{-\infty}^{\infty} g(t)\exp(-j2\pi f t)\,dt = A\int_{-T/2}^{T/2} A\exp(-j2\pi f t)\,dt \\
&= -A\frac{1}{j2\pi f}\exp(-j2\pi f t)\Big|_{t=-T/2}^{t=T/2} \\
&= -\frac{A}{j2\pi f}\left[-\exp(-j2\pi f T/2) - \exp(j2\pi f T/2)\right] \\
&= \frac{A}{\pi f}\left(\frac{\exp(j\pi f T) - \exp(-j\pi f T)}{2j}\right) \\
&= A\left(\frac{\sin(\pi f T)}{\pi f}\right) \\
&= AT\left(\frac{\sin(\pi f T)}{\pi f T}\right) \\
&= AT\mathrm{sinc}(\pi f T)
\end{aligned}
$$

Sinc Function

- Definition

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

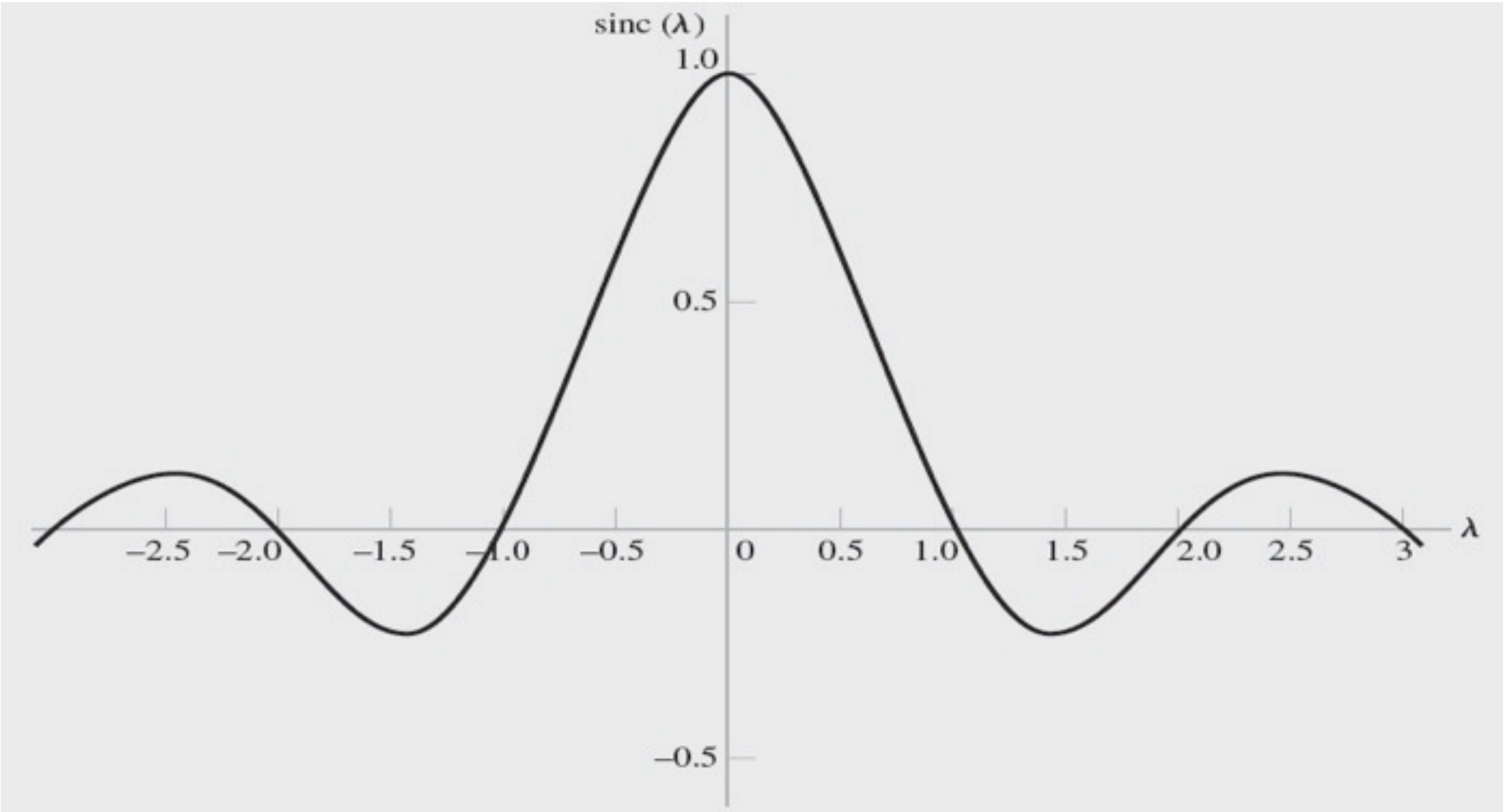


FIGURE 2.3 The sinc function.

[Ref: Haykin & Moher, Textbook]

Fourier Transform of Rectangular Pulse

- Rectangular pulse with the width of T and the height of A so that the area is at the center of zero

$$A\prod\left(\frac{t}{T}\right) \longleftrightarrow AT\mathrm{sinc}(fT)$$

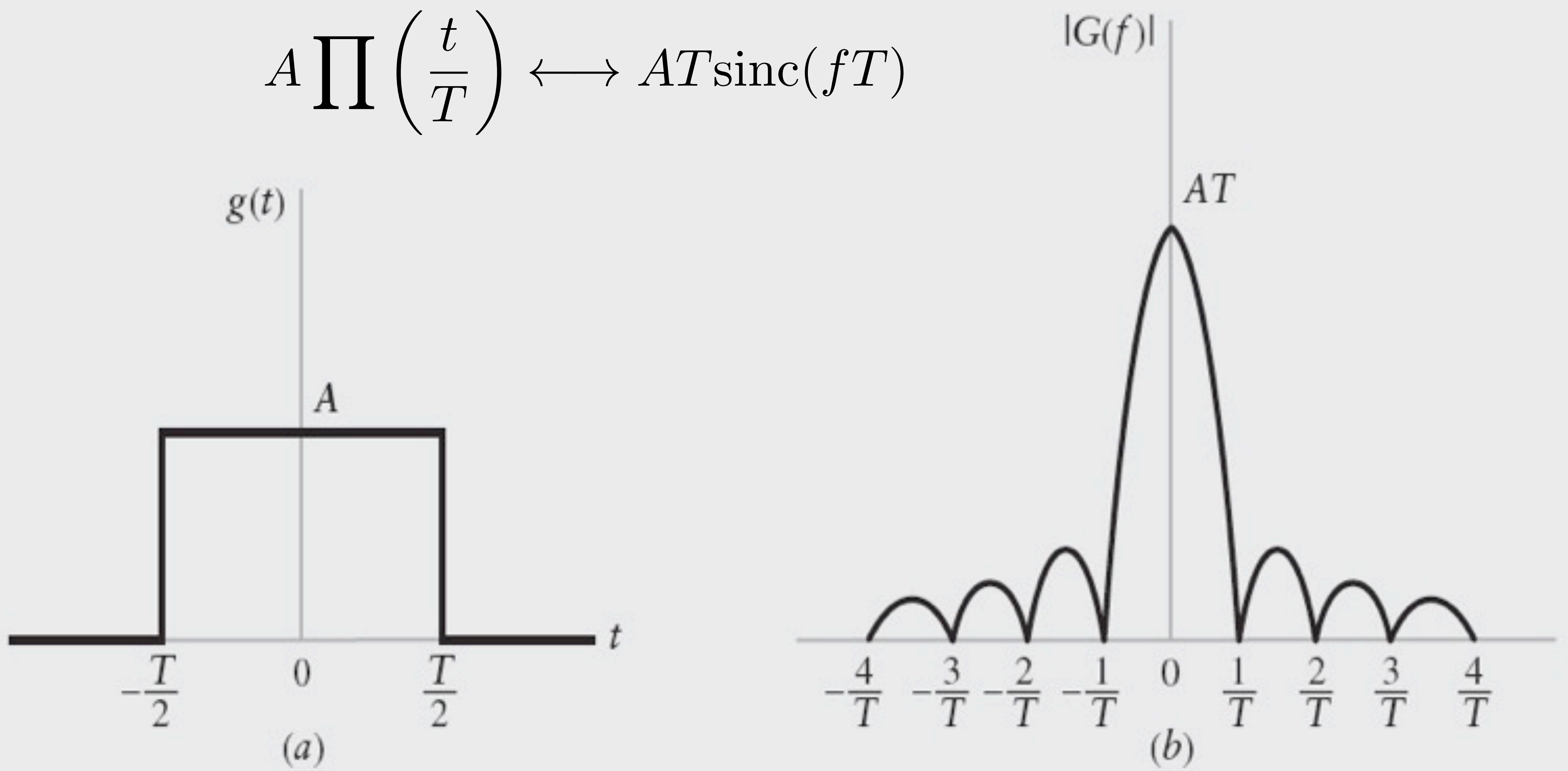


FIGURE 2.2 (*a*) Rectangular pulse. (*b*) Amplitude spectrum.

[Ref: Haykin & Moher, Textbook]

Fourier Transform of Exponential Function

- Exponential function such as

$$g(t) = \exp(-\alpha t)u(t)$$

- Fourier transform

$$\begin{aligned} G(f) &= \mathcal{F}\left[g(t)\right] \\ &= \int_{-\infty}^{\infty} g(t)e^{-j2\pi f t}\, dt = \int_{0}^{\infty} e^{-(\alpha + j2\pi f)t}\, dt \\ &= \frac{1}{\alpha + j2\pi f} \end{aligned}$$

- Similarly for $g(t) = \exp(+\alpha t)u(-t)$

$$G(f) = \frac{1}{\alpha - j2\pi f}$$

Properties of Fourier Transform

- Linearity
- Dilation
- Conjugation rule
- Duality property
- Time shifting property
- Frequency shifting property
- Area property
- Differentiation in the time domain
- Modulation theorem
- Convolution theorem
- Correlation theorem
- Rayleigh's Energy theorem (or Parserval's theorem)

Properties of the Fourier Transform

- Linearity (Superposition) property

$$\text{Let } g_1(t) \Leftrightarrow G_1(f) \text{ and } g_2(t) \Leftrightarrow G_2(f)$$

then for all constants c_1 and c_2

$$c_1 g_1(t) + c_2 g_2(t) \longleftrightarrow c_1 G_1(f) + c_2 G_2(f)$$

- Dilation property

$$g(at) \longleftrightarrow \frac{1}{|a|} G\left(\frac{f}{a}\right)$$

- Proof:

$$\begin{aligned}
\mathcal{F}\left[g(at)\right] &= \int_{-\infty}^{\infty} g(at) \exp(-j2\pi f t)\, dt \\
& \quad \text{change of variable:} \tau = at \\
&= \frac{1}{a} \int_{-\infty}^{\infty} g(\tau) \exp\left[-j2\pi \left(\frac{f}{a}\right) \tau\right] d\tau \\
&= \frac{1}{a} G\left(\frac{f}{a}\right)
\end{aligned}$$

- Reflection property

$$g(-t) \longleftrightarrow G(-f)$$

EXAMPLE 2.3 Combinations of Exponential Pulses

Consider a *double exponential pulse* (defined by (see Fig. 2.6(*a*))

$$g(t) = \begin{cases} \exp(-at), & t > 0 \\ 1, & t = 0 \\ \exp(at), & t < 0 \end{cases}$$

$$= \exp(-a|t|) \qquad (2.15)$$

This pulse may be viewed as the sum of a truncated decaying exponential pulse and a truncated rising exponential pulse. Therefore, using the linearity property and the Fourier-transform pairs of Eqs. (2.12) and (2.13), we find that the Fourier transform of the double exponential pulse of Fig. 2.6(*a*) is

$$G(f) = \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f}$$

$$= \frac{2a}{a^2 + (2\pi f)^2}$$

We thus have the following Fourier-transform pair for the double exponential pulse of Fig. 2.6(*a*):

$$\exp(-a|t|) \rightleftharpoons \frac{2a}{a^2 + (2\pi f)^2} \qquad (2.16)$$

[Ref: Haykin & Moher, Textbook]

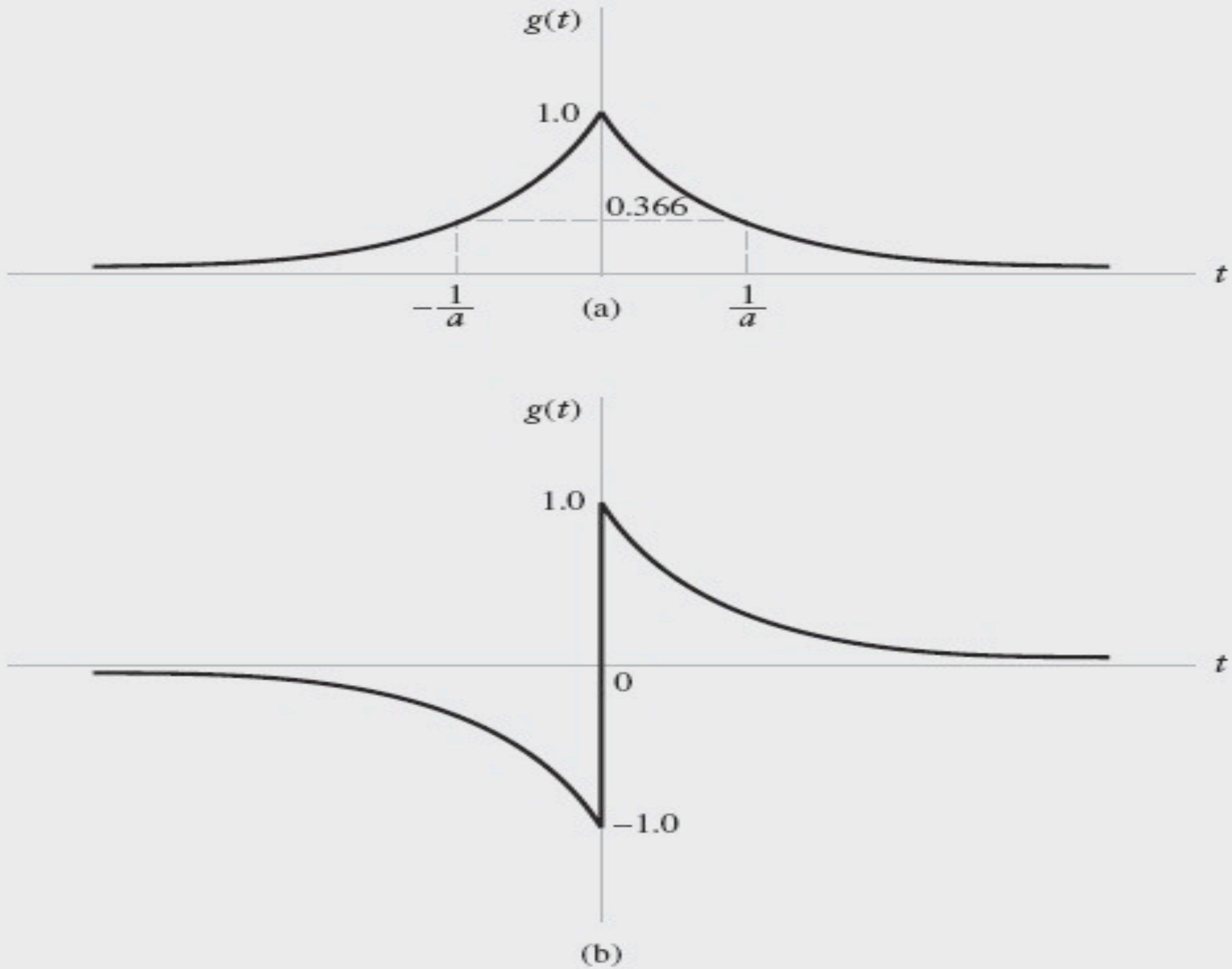


FIGURE 2.6 (*a*) Double-exponential pulse (symmetric). (*b*) Another double-exponential pulse (odd-symmetric).

Note that because of the symmetry in the time domain, as in Fig. 2.6(*a*), the spectrum is real and symmetric; this is a general property of such Fourier-transform pairs.

Another interesting combination is the difference between a truncated decaying exponential pulse and a truncated rising exponential pulse, as shown in Fig. 2.6(*b*). Here we have

$$g(t) = \begin{cases} \exp(-at), & t > 0 \\ 0, & t = 0 \\ -\exp(at), & t < 0 \end{cases} \tag{2.17}$$

We may formulate a compact notation for this composite signal by using the *signum function* that equals +1 for positive time and −1 for negative time, as shown by

$$\operatorname{sgn}(t) = \begin{cases} +1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases} \tag{2.18}$$

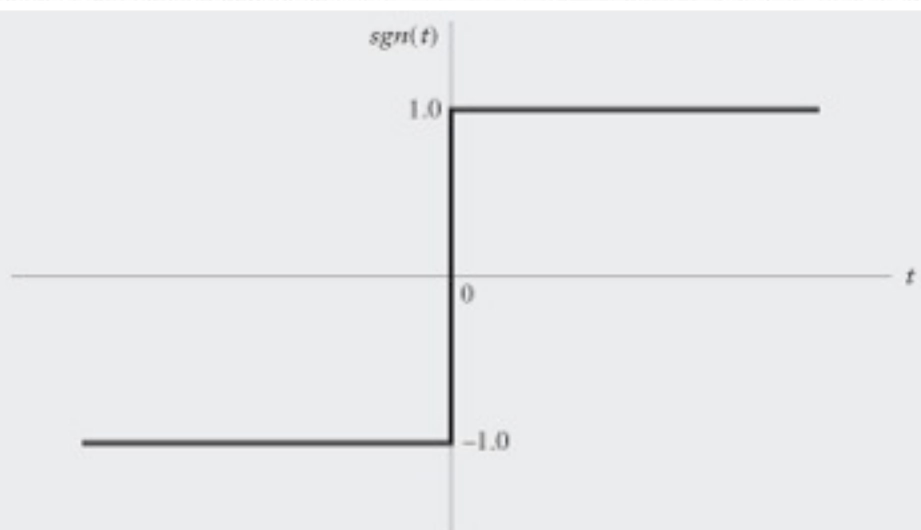


FIGURE 2.7 Signum function.

The signum function is shown in Fig. 2.7. Accordingly, we may reformulate the composite signal $g(t)$ defined in Eq. (2.17) simply as

$$g(t) = \exp(-a|t|)\ \mathrm{sgn}(t)$$

Hence, applying the linearity property of the Fourier transform, we readily find that in light of Eqs. (2.12) and (2.13), the Fourier transform of the signal $g(t)$ is given by

$$\begin{aligned} \mathrm{F}[\exp(-a|t|)\ \mathrm{sgn}(t)] &= \frac{1}{a + j2\pi f} - \frac{1}{a - j2\pi f} \\ &= \frac{-j4\pi f}{a^2 + (2\pi f)^2} \end{aligned}$$

We thus have the Fourier-transform pair

$$\exp(-a|t|)\ \mathrm{sgn}(t) \rightleftharpoons \frac{-j4\pi f}{a^2 + (2\pi f)^2} \tag{2.19}$$

In contrast to the Fourier-transform pair of Eq. (2.16), the Fourier transform in Eq. (2.19) is odd and purely imaginary. It is a general property of Fourier-transform pairs that apply to an *odd-symmetric* time function, which satisfies the condition $g(-t) = -g(t)$, as in Fig. 2.6(b); such a time function has an odd and purely imaginary function as its Fourier transform.

- Conjugation rule

Let $g(t) \longleftrightarrow G(f)$, then for a complex-valued time function $g(t)$

$$g^*(t) \longleftrightarrow G^*(-f)$$

Prove this:

$$g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi ft)df$$

$$g^*(t) = \int_{-\infty}^{\infty} G^*(f)\exp(-j2\pi ft)df$$

$$g^*(t) = -\int_{\infty}^{-\infty} G^*(-f)\exp(j2\pi ft)df$$

$$= \int_{-\infty}^{\infty} G^*(-f)\exp(j2\pi ft)df$$

$$g^*(-t) \longleftrightarrow G^*(f)$$

- Duality property

If $g(t) \longleftrightarrow G(f)$, then $G(t) \longleftrightarrow g(-f)$

$$g(-t) = \int_{-\infty}^{\infty} G(f) \exp(-j2\pi f t)\, df$$

which can be expanded part in going from the time domain to the frequency domain:

$$g(-f) = \int_{-\infty}^{\infty} G(t) \exp(-j2\pi ft) dt$$

Example of Dual Property: Sinc Pulse

- We have the following pair of the Fourier transform:

$$g(t) = A\,\mathrm{sinc}(2Wt) \longleftrightarrow G(f) = \frac{A}{2W}\mathrm{rect}\left(\frac{f}{2W}\right)$$

- Then, if the time function, given as

$$h(t) = G(t) = \frac{A}{2W}\mathrm{rect}\left(\frac{t}{2W}\right) \longleftrightarrow H(f) = g(-f) = A\,\mathrm{sinc}(-2Wf) = A\,\mathrm{sinc}(2Wf)$$

- Time shifting property

If $g(t) \longleftrightarrow G(f)$, then $g(t - t_0) \longleftrightarrow G(f) \exp(-j2\pi f t_0)$

- Frequency shifting property

If $g(t) \longleftrightarrow G(f)$, then $\exp(j2\pi f_c t) g(t) \longleftrightarrow G(f - f_c)$

- Area property under $g(t)$

If $g(t) \longleftrightarrow G(f)$, then

$$\int_{-\infty}^{\infty} g(t)\, dt = G(0)$$

Example of Frequency Shifting Property

- Find the FT of radio frequency pulse given as

$$g(t) = \mathrm{rect}\left(\frac{t}{T}\right)\cos(2\pi f_c t)$$

Using the Euler's formula we have

$$\cos(2\pi f_c t) = \frac{1}{2}\left[\exp(2\pi f_c t) + \exp(-j2\pi f_c t)\right]$$

Then using the frequency shifting property of the Fourier transform we get the desired result:

$$\mathrm{rect}\left(\frac{t}{T}\right)\cos(2\pi f_c t) \longleftrightarrow \frac{T}{2}\left\{\mathrm{sinc}[T(f-f_c)] + \mathrm{sinc}[T(f+f_c)]\right\}$$

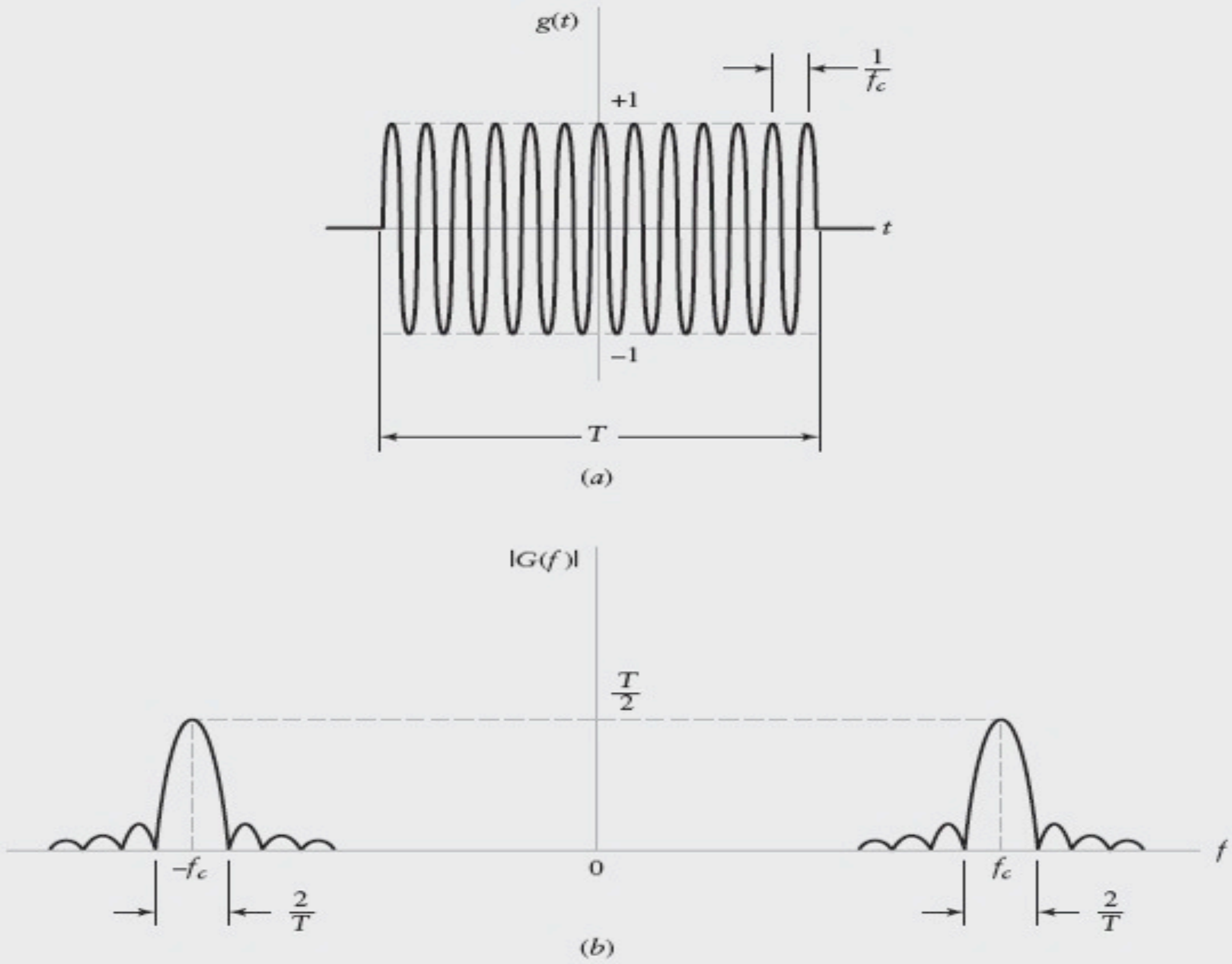


FIGURE 2.9 (a) RF pulse of unit amplitude and duration T. (b) Amplitude spectrum.

[Ref: Haykin & Moher, Textbook]

- In the special case of $f_c T >> 1$, that is, the frequency f_c is large compared to the reciprocal of the pulse duration T - we may use the approximate result

$$G(f) = \begin{cases} \frac{T}{2}\text{sinc}[T(f - f_c)], & f > 0 \\ 0, & f = 0, \\ \frac{T}{2}\text{sinc}[T(f + f_c)], & f < 0 \end{cases}$$

- Area property under $G(f)$

$$g(0) = \int_{-\infty}^{\infty} G(f)\, df$$

- Differentiation in the time domain

If $g(t) \longleftrightarrow G(f)$, then

$$\frac{d}{dt} g(t) \longleftrightarrow j2\pi f G(f)$$

and

$$\frac{d^n}{dt^n} g(t) \longleftrightarrow (j2\pi f)^n G(f)$$

- Modulation theorem

Let $g_1(t) \longleftrightarrow G_1(f), \text{ and } g_2(t) \longleftrightarrow G_2(f)$, then

$$g_1(t)g_2(t) \longleftrightarrow \int_{-\infty}^{\infty} G_1(\lambda)G_2(f-\lambda)\, d\lambda$$

- Convolution Theorem

$$\int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)\, d\tau \longleftrightarrow G_1(f)G_2(f)$$

$$\downarrow$$

$$g_1(t) * g_2(t) \longleftrightarrow G_1(f)G_2(f)$$

- Correlation theorem

$$\int_{-\infty}^{\infty} g_1(\tau) g_2^*(t-\tau)\, d\tau \longleftrightarrow G_1(f) G_2^*(f)$$

- Rayleigh's Energy theorem

$$\int_{-\infty}^{\infty} |g(t)|^2\, dt = \int_{-\infty}^{\infty} |G(f)|^2\, df$$

EXAMPLE 2.9 Sinc Pulse (continued)

Consider again the sinc pulse $A\ sinc\ (2Wt)$. The energy of this pulse equals

$$E = A^2 \int_{-\infty}^{\infty} \text{sinc}^2(2Wt)\, dt$$

The integral in the right-hand side of this equation is rather difficult to evaluate. However, we note from Example 2.4 that the Fourier transform of the sinc pulse $A\ \text{sinc}(2Wt)$ is equal to $(A/2W)\ \text{rect}(f/2W)$; hence, applying Rayleigh's energy theorem to the problem at hand, we readily obtain the desired result:

$$\begin{aligned} E &= \left(\frac{A}{2W}\right)^2 \int_{-\infty}^{\infty} \text{rect}^2\left(\frac{f}{2W}\right) df \\ &= \left(\frac{A}{2W}\right)^2 \int_{-W}^{W} df \\ &= \frac{A^2}{2W} \end{aligned} \tag{2.57}$$

[Ref: Haykin & Moher, Textbook]