

Communication Systems II

[KECE322_01]

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Lecture #15

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Outline

- Average symbol rate
 - M-ary orthogonal signals
- A Union bound on the probability of error

Probability of Error for M-ary Orthogonal Signals

- Each of M-ary orthogonal signals has equal energy.
- For equal energy orthogonal signals, the optimum detector selects the signal resulting in the largest cross correlation between the received vector \mathbf{y} and each of the M possible transmitted signal vectors $\{\mathbf{s}_m\}$, i.e.,

$$C(\mathbf{y}, \mathbf{s}_m) = \mathbf{y} \cdot \mathbf{s}_m = \sum_{k=1}^M y_k s_{mk}, \quad m = 1, 2, \dots, M.$$

- To evaluate the probability of error, let us assume that the signal \mathbf{s}_1 is transmitted. Then the vector at the input to the detector is

$$\mathbf{y} = \left(\sqrt{E_s} + n_1, n_2, n_3, \dots, n_M \right),$$

- where $n_1, n_2, n_3, \dots, n_M$ are zero mean, mutually statistically independent Gaussian random variables with equal variance $N_0/2$.

■ Cross-correlation metric

$$\begin{aligned} C(\mathbf{y}, \mathbf{s}_1) &= \sqrt{\mathcal{E}_s}(\sqrt{\mathcal{E}_s} + n_1); \\ C(\mathbf{y}, \mathbf{s}_2) &= \sqrt{\mathcal{E}_s}n_2; \\ &\vdots \\ C(\mathbf{y}, \mathbf{s}_M) &= \sqrt{\mathcal{E}_s}n_M. \end{aligned}$$

- Note that we can eliminate the scale factor $\sqrt{\mathcal{E}_s}$ for the comparisons.

■ PDF of the first correlator output with the elimination of $\sqrt{\mathcal{E}_s}$.

$$f(y_1) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(y_1 - \sqrt{\mathcal{E}_s})^2}{N_0}}$$

- PDF's of the other M-1 correlator outputs

$$f(y_m) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{y_m^2}{N_0}}, \quad m = 2, 3, \dots, M$$

■ Correct decision probability when $s_1(t)$ is transmitted

$$P_{c|s_1} = \int_{-\infty}^{\infty} \Pr[n_2 < y_1, n_3 < y_1, \dots, n_M < y_1 | s_1] f_{y_1}(y_1) dy_1$$

● For equally probable case,

$$P_c = \int_{-\infty}^{\infty} \Pr(n_2 < y_1, n_3 < y_1, \dots, n_M < y_1 | y_1) f_{y_1}(y_1) dy_1$$

● Note that

$$\begin{aligned} \Pr(n_M < y_1 | y_1) &= \int_{-\infty}^{y_1} f(y_m) dy_m, \quad m = 2, 3, \dots, M \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\sqrt{2y_1^2}}{N_0}} e^{-\frac{y_m^2}{2}} dy_m \\ &= 1 - Q\left(\sqrt{\frac{2y_1^2}{N_0}}\right). \end{aligned}$$

- Probability of correct decision

$$P_c = \int_{-\infty}^{\infty} \left[1 - Q \left(\sqrt{\frac{2y_1^2}{N_0}} \right) \right]^{M-1} f_{y_1}(y_1) dy_1$$

- Probability of symbol error (Symbol error rate)

$$P_M = 1 - \int_{-\infty}^{\infty} \left[1 - Q \left(\sqrt{\frac{2y_1^2}{N_0}} \right) \right]^{M-1} \frac{1}{\sqrt{\pi N_0}} e^{-(y-\sqrt{\mathcal{E}_s})^2/N_0} dy_1$$

Change of variable $x = \frac{2y_1^2}{N_0}$

Then we have

$$\begin{aligned} P_M &= 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [1 - Q(x)]^{M-1} e^{-(x-\sqrt{2\mathcal{E}_s/N_0})} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ 1 - [1 - Q(x)]^{M-1} \right\} e^{-(x-\sqrt{2\mathcal{E}_s/N_0})} dx \end{aligned}$$

■ Average bit error rate

$$P_b = \frac{1}{k} \sum_{n=1}^k n \binom{n}{k} \frac{P_M}{2^k - 1} = \frac{2^{k-1}}{2^k - 1} P_M \approx \frac{P_M}{2}, \quad k \gg 1.$$

■ Symbol error rate

A Union Bound on the Probability of Error

- BER of binary antipodal signals

$$P_2 = Q\left(\frac{d}{\sqrt{2N_0}}\right) \quad \text{where} \quad d^2 = \int_{-\infty}^{\infty} (s_1(t) - s_2(t))^2 dt$$

- SER of M-ary PAM

$$P_M = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6(\log_2 M)\mathcal{E}_{bav}}{(M^2-1)N_0}}\right).$$

- A Union bound

$$P_m = P(\text{error} | s_m(t) \text{ sent}) = P\left(\bigcup_{i=1, i \neq m}^M \left| s_m(t) \text{ sent} \right.\right) \leq \sum_{i=1, i \neq m} P(E_i | s_m(t) \text{ sent}).$$

where E_i : event that message i is detected at the receiver

- Necessary condition for $s_i(t)$ to be detected at the receiver when signal $s_m(t)$ is sent that y closer than to s_i than to s_m :

$$P(E_i | s_m(t) \text{ sent}) \leq P[D(\mathbf{y}, \mathbf{s}_i) < D(\mathbf{y}, \mathbf{s}_m)] \\ = Q\left(\frac{d_{mi}}{\sqrt{2N_0}}\right)$$

$$\Rightarrow P(E_i | s_m(t) \text{ sent}) \leq Q\left(\frac{d_{mi}}{\sqrt{2N_0}}\right)$$

- Hence, we have an upper bound

$$P_m \leq \sum_{i=1, i \neq m}^M Q\left(\frac{d_{mi}}{\sqrt{2N_0}}\right).$$

- Let us define the minimum distance as

$$d_{\min} = \min_{\substack{1 \leq m, m' \leq M \\ m' \neq m}} d_{mm'}$$

- Thus for any m , $d_{mi} \geq d_{\min}$, so that we have

$$Q\left(\frac{d_{mi}}{\sqrt{2N_0}}\right) \leq Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

- Hence, we obtain an upper bound as

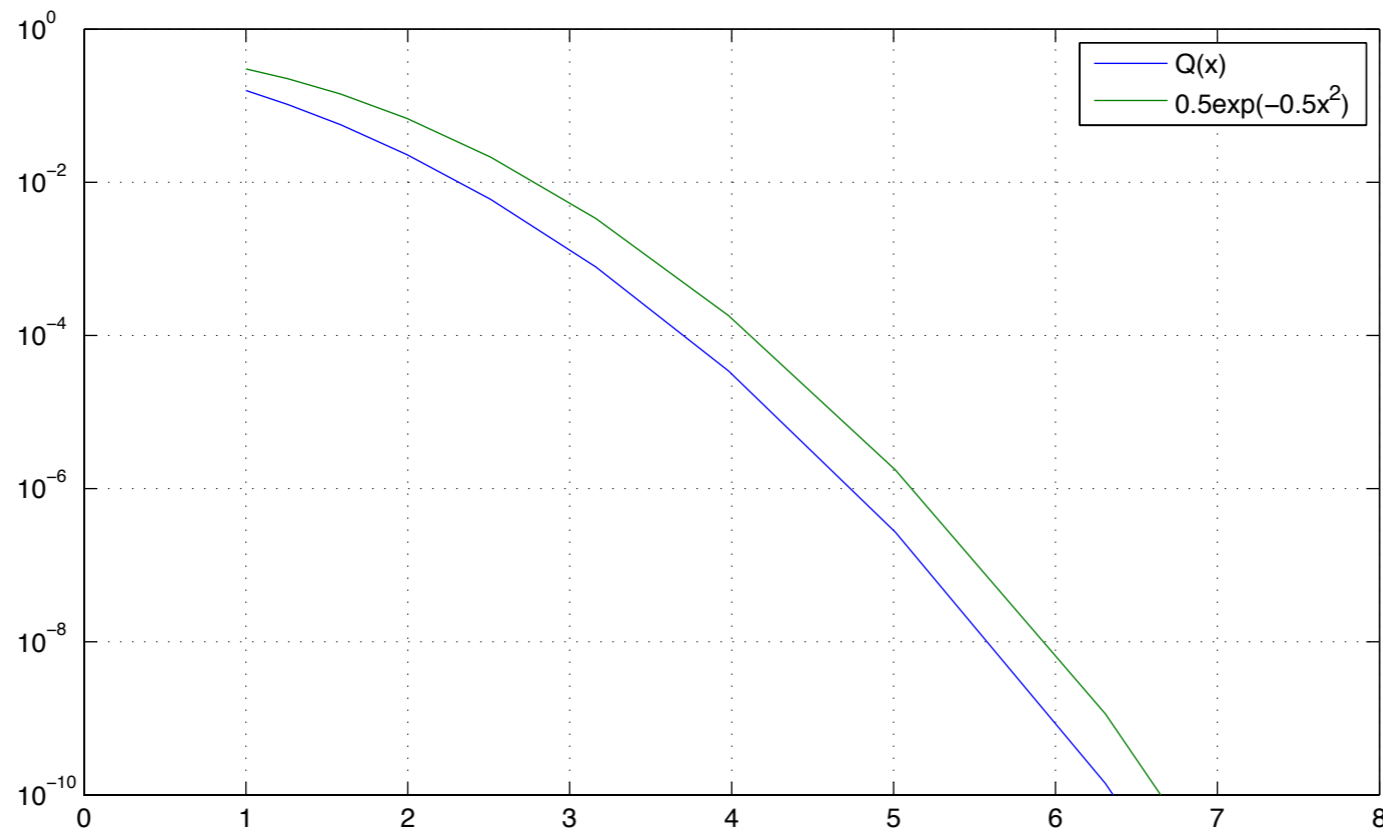
$$P_m \leq \sum_{i=1, i \neq m}^M Q\left(\frac{d_{mi}}{\sqrt{2N_0}}\right) = (M-1)Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

- Also note that

$$Q(x) \leq \frac{1}{2}e^{-\frac{x^2}{2}}$$

- Therefore, we can rewrite the upper bound as

$$P_m \leq \sum_{i=1, i \neq m}^M Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) = (M-1)Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) \leq \frac{M-1}{2} e^{-\frac{d_{min}^2}{4N_0}}$$



■ A union bound on the error probability of M-ary orthogonal signaling

- M-ary orthogonal signals are equidistant with

$$d_{mm'}^2 = \|\mathbf{s}_m - \mathbf{s}_{m'}\|^2 = 2\mathcal{E}_s$$

- ◆ Therefore,

$$d_{\min} = \sqrt{2\mathcal{E}_s}.$$

- Union bound

$$P_M \leq \frac{M-1}{2} e^{-\frac{\mathcal{E}_s}{2N_0}} \leq M e^{-\frac{\mathcal{E}_s}{2N_0}}.$$

- Using $M = 2^k$ and $\mathcal{E}_s = k\mathcal{E}_b$, we have

$$P_M \leq 2^k e^{-k\mathcal{E}_b/2N_0} = e^{-k(\mathcal{E}_b/N_0 - 2 \ln 2)/2}.$$

- As $k \rightarrow \infty$ or $M \rightarrow \infty$, the probability of error approaches zero exponentially, provided that \mathcal{E}_b/N_0 is greater than $2 \ln 2$, i.e.,

$$\frac{\mathcal{E}_b}{N_0} > 2 \ln 2 = 1.39 \approx 1.42 \text{ dB}$$

- ◆ The simple upper bound on the probability of error implies that as long as $\text{SNR} > 1.42 \text{ dB}$, we can achieve an arbitrary low P_M .
- ◆ However, this union bound is not a very tight upper bound at low SNR values.
- ◆ In fact, by more elaborate bounding techniques, it can be shown that $P_M \rightarrow 0$ as $k \rightarrow \infty$ provided that

$$\frac{\mathcal{E}_b}{N_0} > \ln 2 = 0.693 \approx -1.6 \text{ dB}$$

- Hence, -1.6 dB is the minimum required SNR/bit to achieve an arbitrarily small probability of error in the limit as $k \rightarrow \infty$ ($M \rightarrow \infty$).
- ◆ This minimum SNR/bit (-1.6 dB) is called the *Shannon limit* for an additive white Gaussian noise channel.