Communication Systems II

[KECE322_01]

<2012-2nd Semester>

Lecture #2

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Outline

- Review of probability and random variables (Secs. 5.1.1 - 5.1.4)
  - sample space, events, and probability
  - conditional probability
  - random variables
  - functions of random variable

Experiment, Outcome, and Sample space

random experiment	possible outcomes
•flipping a coin	• H or T
•drawing a card from a deck of cards	• one of 52 cards
•throwing a die	• 1,2,3,4,5,6

- Sample space Ω
  - the set of all possible outcomes

$$\Omega = \{H,\ T\}$$

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

  - Denote an outcome as ω , then $\omega \in \Omega$

Continuous vs. Discrete Sample Space

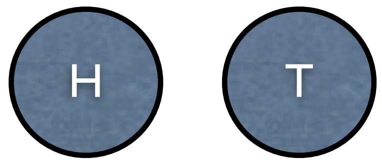
- Continuous sample space
  - Received signal
  - Temperature
- Discrete sample space
  - Flipping a coin
  - bits generated from the source

Events

- Events, E
  - subsets of the sample space
  - Example: In the experiment of throwing a die,
    - the event "the outcome is odd" consists of outcomes 1, 3, and 5.
    - the event "the outcome is greater than 3" consists of outcomes 4, 5 and 6.
    - the event "the outcome divides 4" consists of the single outcome 4.
  - Example: In the experiment of picking a number between 0 and 1,
    - we can define an event as "the outcome is less than 0.7", "the outcome is between 0.2 and 0.5", "the outcome is 0.5".
  - Events are disjoint if their intersection is empty
    - In throwing a die, the events "the outcome is odd" and "the outcome divides 4" are disjoint.

Intuitive Concept of Probability

- *Experiment*: Flipping a coin

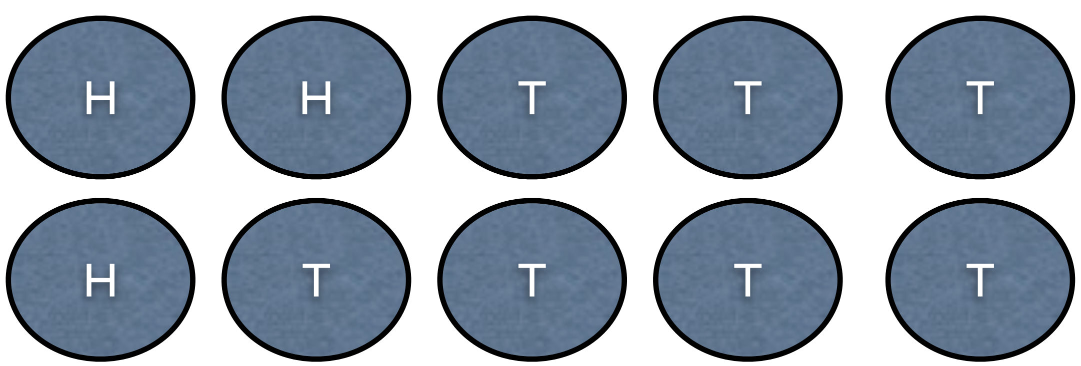


  - Head or Tail

- What is the probability of 'Head' or 'Tail' to occur in the event of flipping a coin?

$$P(H) = \frac{1}{2}, \qquad P(T) = \frac{1}{2}$$

- *Experiment*: Flipping a coin 10 times

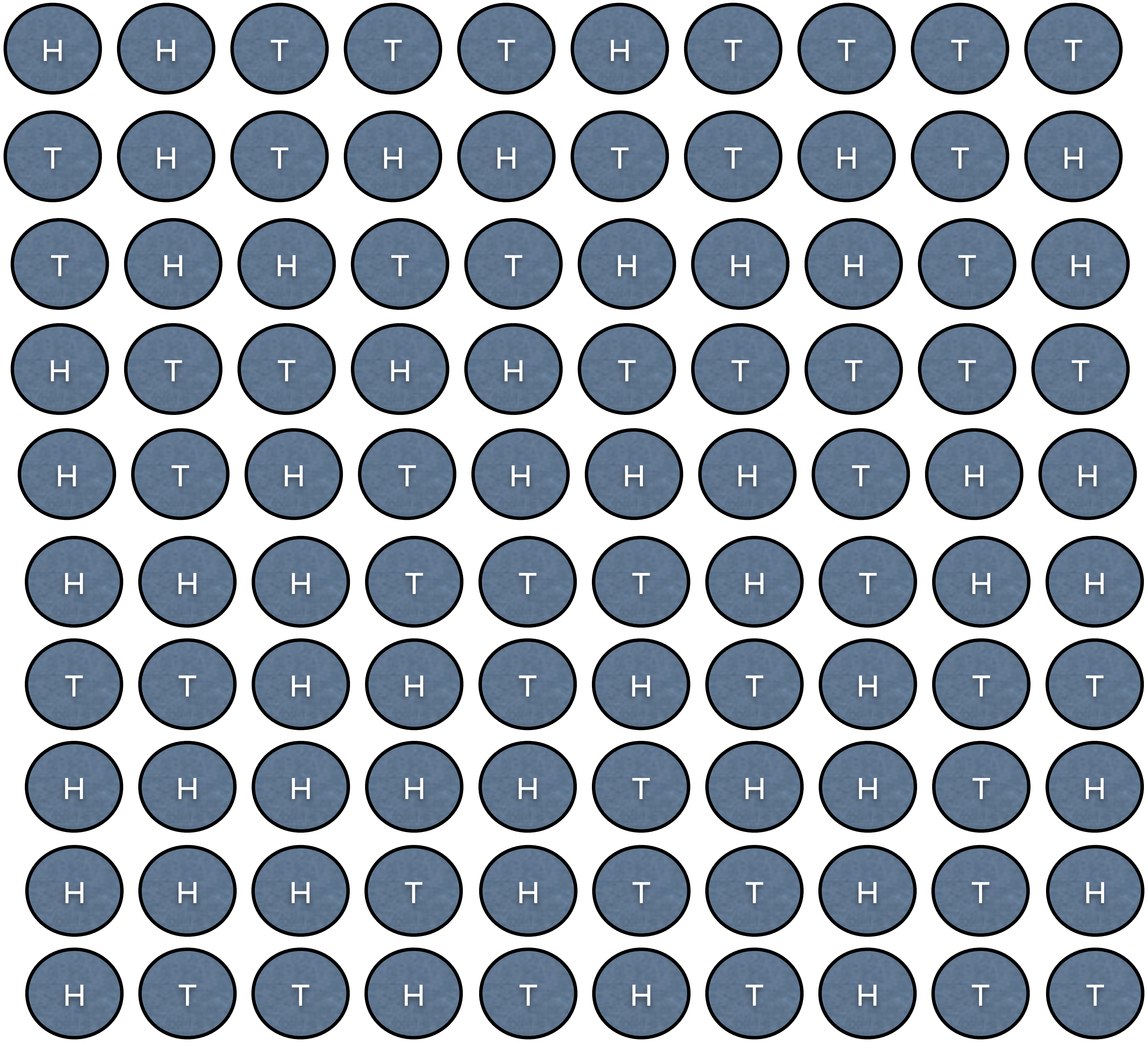


Did you get 1/2 of probability for 'H' or 'T'?

$$P(H) = \frac{3}{10}$$

$$P(T) = \frac{7}{10}$$

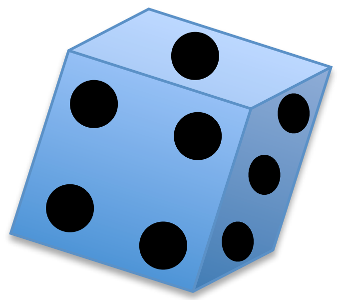
Experiment: Flipping a coin 100 times



$$P(H) = \frac{51}{100}$$

$$P(T) = \frac{49}{100}$$

- *Experiment:* Tossing a die



  - Possible outcomes=1,2,3,4,5,6

- *Experiment:* Toss a die two times then the total possible outcomes are

$$\begin{aligned}\{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\&(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\&(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\&(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\&(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\&(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}\end{aligned}$$

→ 36 possible outcomes

- Toss a die two times then the total possible outcomes of the sum are

$$\{2,3,4,5,6,7,8,9,10,11,12\}$$

- In the experiment of tossing a die two times and observing the sum more than 8

$$\{(3,6),(4,5),(4,6),(5,4),(5,5),(5,6),(6,3),(6,4),(6,5),(6,6)\}$$

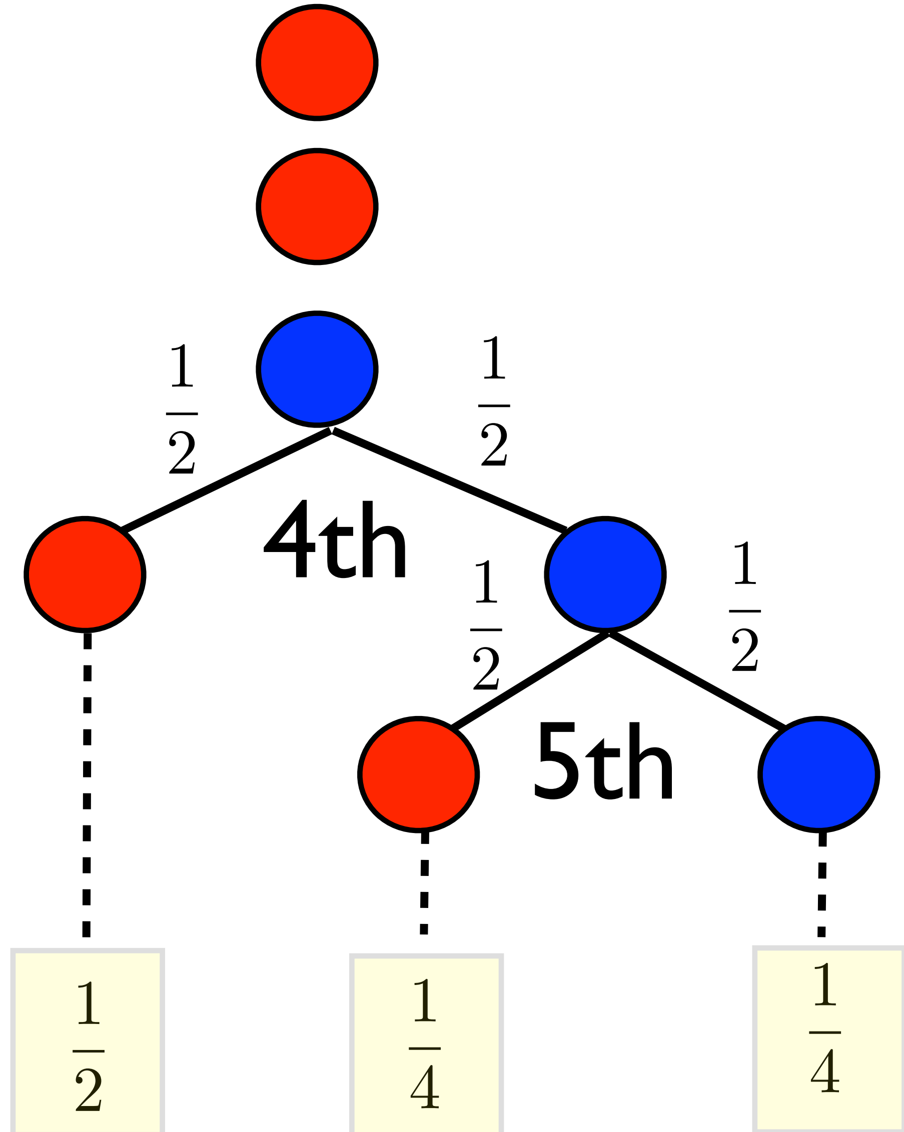
  - 10 outcomes out of 36 possible outcomes

- Now what do you say about the probability of the ***event*** that the sum is more than 8?

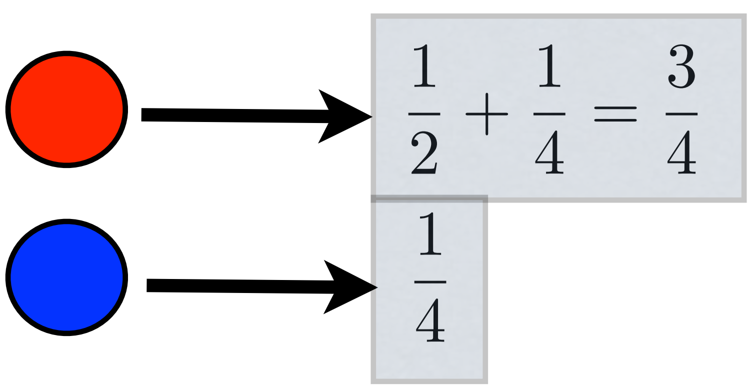
$$P(\text{event of more than } 8) = \frac{10}{36}$$

Origin of the Probability Theory: Gambling and Probability

Blaise Pascal (1623-1662)



Pierre de Fermat (1601-1665)



- Gambler, Chevalier de Mere
  - Mere's questions:
    - Two gamblers, A and B, are gambling. The game rule is that one who wins the three times wins the game.
    - How do we can distribute the money if the game is sopped and A won 2 times and B won one time?

Probability

- We define a probability P as a set of function assigning nonnegative values to all events E such that the following conditions are satisfied:

  1. $0 \leq P(E) \leq 1$ for all events
  2. $P(\Omega) = 1$.
  3. For disjoint events $E_1,\ E_2,\ E_3, \cdots$ (i.e., events for which $E_i \cap E_j \neq \phi$ for all $i \neq j$, where ϕ is the empty set), we have $P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$.

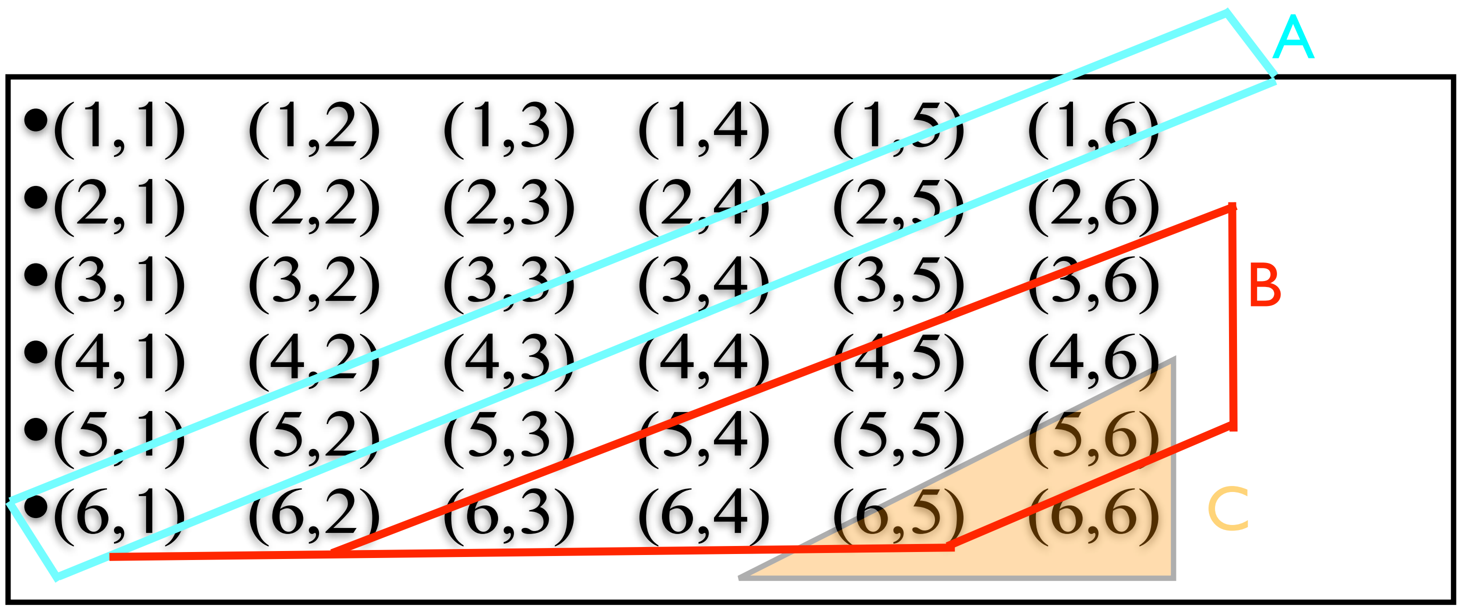
- Law of large number and definition of probability

$$\lim_{n\to\infty} \left(\frac{\text{number of the occurrence of event } A}{\text{number of experiments, } n} \right) = P(A)$$

- We can also define the probability such as

$$P(A) = \frac{\text{length of event}}{\text{length of sample space}}$$

Example of tossing two dies



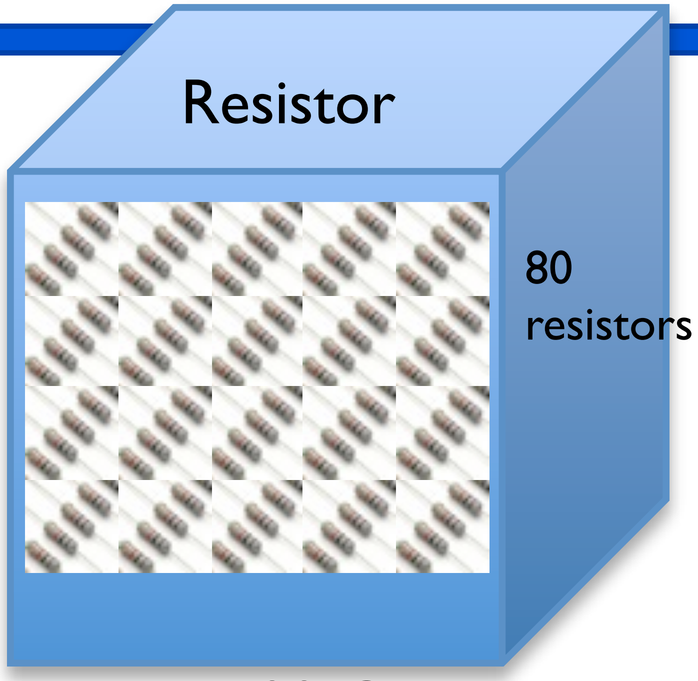
$$A = \{\text{sum=7}\},\ B = \{8 < \text{sum} \leq 11\},\ \text{and},\ C = \{10 < \text{sum}\}$$

- events $A_{ij} = \{\text{sum for outcome } (i,j) = i + j\}$

$$P(A) = P\left(\bigcup_{i=1}^{6} A_{i,7-i}\right) = \sum_{i=1}^{6} P(A_{i,7-i}) = 6\left(\frac{1}{36}\right) = \frac{1}{6}$$

$$P(B) = 9\left(\frac{1}{36}\right) = \frac{1}{4} \qquad P(C) = 3\left(\frac{1}{36}\right) = \frac{1}{12}$$

Example



- In a box there are 80 resistors with the same size and shape.
  - 18 are 10 Ohm
  - 12 are 22 Ohm
  - 33 are 27 Ohm
  - 17 are 47 Ohm
- Experiment: randomly draw out one resistor from the box with each one being "equally likely" to be drawn.

$$P(\text{draw } 10\Omega) = 18/80, \quad P(\text{draw } 22\Omega) = 12/80$$
$$P(\text{draw } 27\Omega) = 33/80, \quad P(\text{draw } 47\Omega) = 17/80$$

- Suppose a 22 Ohm resistor is drawn from the box and not replaced. A second resistor is then drawn from the box.
  - In a box there are 80 resistors with the same size and shape, we have for the second drawing

$$P(\text{draw } 10\Omega|22\Omega) = 18/79, \quad P(\text{draw } 22\Omega|22\Omega) = 12/79$$
$$P(\text{draw } 27\Omega|22\Omega) = 33/79, \quad P(\text{draw } 47\Omega|22\Omega) = 17/79$$

⇒ *Conditional probability*

Joint Probability

- Joint probability for two events A and B

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

- Equivalently

$$P(A \cup B) = P(A) + P(B) - P(A \cup B) \leq P(A) + P(B)$$

- Mutually exclusive events if $A \cap B = \phi$, and therefore, $P(A \cap B) = P(\phi) = 0$

Conditional Probability

- Given some event B with nonzero probability $P(B) > 0$ we define the conditional probability of an event A, given B, by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

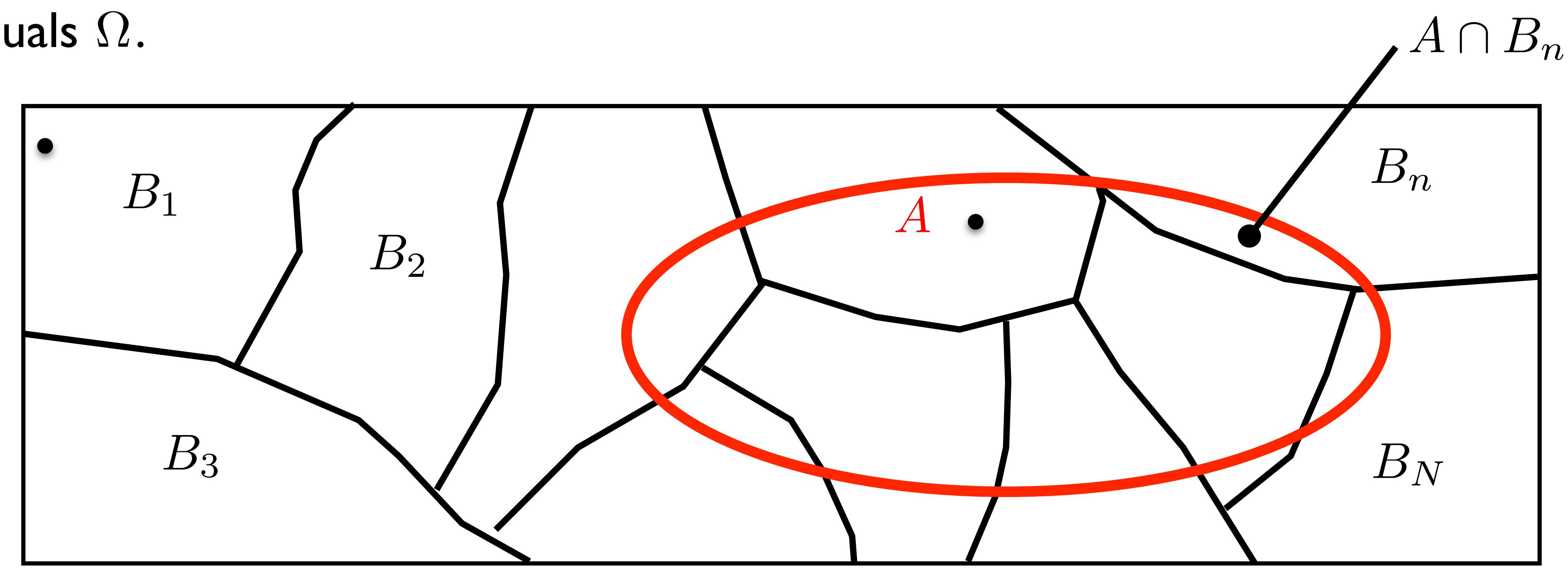
  - The probability $P(A|B)$ simply reflects the fact that the probability of an event A may depend on a second event B.
  - If A and B are mutually exclusive, $A \cap B = \phi$, and $P(A|B) = 0$.
  - Conditional probability is a defined quantity and cannot be proven.
    - However, as a probability it must satisfy the three axioms.
    - From axiom 2,

$$P(\Omega|B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

Total Probability

- The probability $P(A)$ of any event A depends on a sample space Ω can be expressed in terms of conditional probabilities.

- Suppose we are given N mutually exclusive events $B_n,\ n = 1, 2, \ldots, N$, whose union equals Ω.



$$P(A) = \sum_{n=1}^{N} P(A \cap B_n) = \sum_{n=1}^{N} P(A|B_n)P(B_n)$$

Bayes' Theorem

- Bayes' theorem

$$P(B_n|A) = \frac{P(B_n \cap A)}{P(A)} = \frac{P(A|B_n)P(B_n)}{P(A)}$$

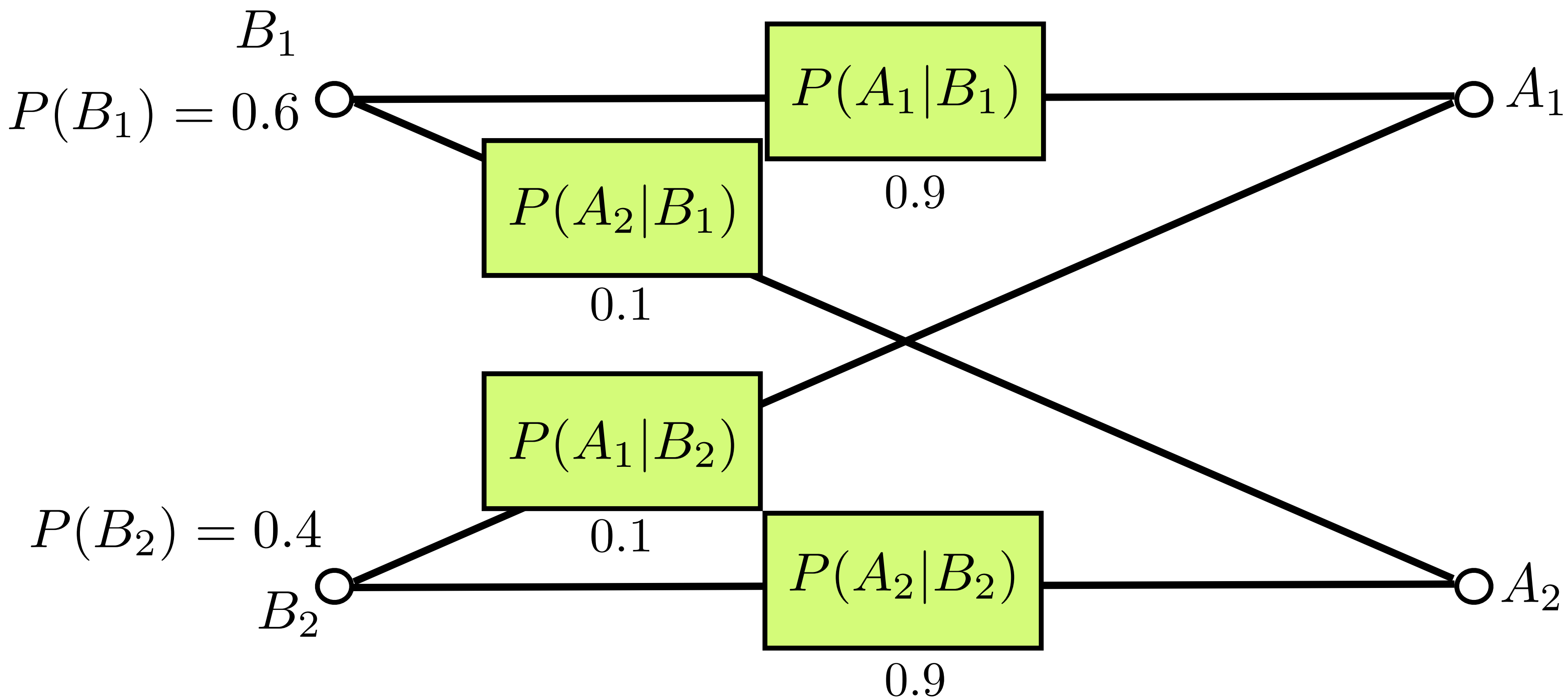
$$P(A|B_n) = \frac{P(A \cap B_n)}{P(B_n)} = \frac{P(B_n|A)P(A)}{P(B_n)}$$

- We can also rewrite

$$\begin{aligned} P(B_n|A) &= \frac{P(A \cap B_n)}{P(A)} = \frac{P(A|B_n)P(B_n)}{P(A)} = \frac{P(A|B_n)P(B_n)}{P(A|B_1)P(B_1) + \cdots + P(A|B_N)P(B_N)} \\ &= \frac{P(A|B_n)P(B_n)}{\sum_{j=1}^{N} P(A|B_j)P(B_j)}. \end{aligned}$$

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

Example: Binary Symmetric Channel (BSC)



$P(A_1)$ and $P(A_2)$?

$P(B_1|A_1)$ and $P(B_2|A_2)$?

$P(B_1|A_2)$ and $P(B_2|A_1)$?

$$\begin{aligned} P(A_1) &= P(A_1|B_1)P(B_1) + P(A_1|B_2)P(B_2) = 0.9(0.6) + 0.1(0.4) = 0.58 \\ P(A_2) &= P(A_2|B_1)P(B_1) + P(A_2|B_2)P(B_2) = 0.1(0.6) + 0.9(0.4) = 0.42 \end{aligned}$$

$$P(B_1|A_1) = \frac{P(A_1|B_1)P(B_1)}{P(A_1)} = \frac{0.9(0.6)}{0.58} = \frac{0.54}{0.58} \approx 0.931$$

$$P(B_2|A_2) = \frac{P(A_2|B_2)P(B_2)}{P(A_2)} = \frac{0.9(0.4)}{0.42} = \frac{0.36}{0.42} \approx 0.857$$

$$P(B_1|A_2) = \frac{P(A_2|B_1)P(B_1)}{P(A_2)} = \frac{0.1(0.6)}{0.42} = \frac{0.06}{0.42} \approx 0.143$$

$$P(B_2|A_1) = \frac{P(A_1|B_2)P(B_2)}{P(A_1)} = \frac{0.1(0.4)}{0.58} = \frac{0.04}{0.58} \approx 0.069$$

Independent Events

- Statistically independent if

$$P(A|B) = P(A) \qquad\qquad P(B|A) = P(B)$$

  - We also have for statistically events

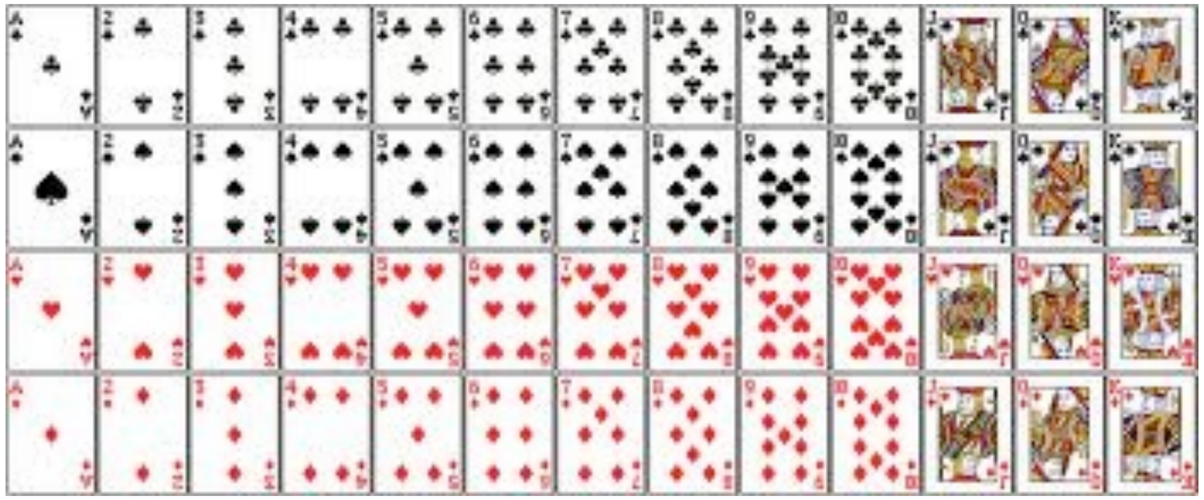
$$P(A \cap B) = P(A|B)P(B) = P(A)P(B)$$

  - If A and B are statistically independent,

$$P(A \cap B) = P(A|B)P(B) = P(A)P(B) \neq 0$$

  - Note
    - If A and B are nonzero probabilities of occurrences and statistically independent,
    - which means $A \cap B \neq \phi$.
  - In order for two events to be independent they must have an intersection $A \cap B \neq \phi$

Example



- Define events as follows:
  - Event A : select a king
  - Event B: select a jack or queen
  - Event C: select a heart

$$P(A) = \frac{4}{52},\ P(B) = \frac{8}{52},\ \text{and } P(C) = \frac{13}{52}$$

- Joint probabilities

$$P(A \cap B) = 0,\ P(A \cap C) = \frac{1}{52},\ P(B \cap C) = \frac{2}{52}$$

- Independent?

$$P(A \cap B) = 0 \quad \neq \quad P(A)P(B) = \frac{32}{52^2}$$

$$P(A \cap C) \quad = \quad \frac{1}{52} = P(A)P(C) = \frac{1}{52}$$

$$P(B \cap C) \quad = \quad \frac{2}{52} = P(B)P(C) = \frac{2}{52}$$

Multiple Independent Events

- Three independents

$$
\begin{aligned}
P(A_1 \cap A_2) &= P(A_1)P(A_2) \\
P(A_1 \cap A_3) &= P(A_1)P(A_3) \\
P(A_2 \cap A_3) &= P(A_2)P(A_3) \\
P(A_1 \cap A_2 \cap A_3) &= P(A_1)P(A_2)P(A_3)
\end{aligned}
$$

Permutation and Combination

- Permutation

$$\text{ordering of } r \text{ elements taken from } n = n(n-1)(n-2)\cdots(n-r+1)$$
$$= \frac{n!}{(n-r)!} = P_r^n \quad r-1,2,\ldots,n$$

- Combination

binomial coefficient

$$r \text{ elements taken from } n = \binom{n}{r} = \frac{n!}{(n-r)!r!} =_n C_r$$

- Binomial coefficient

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

- Symmetry of binomial coefficient

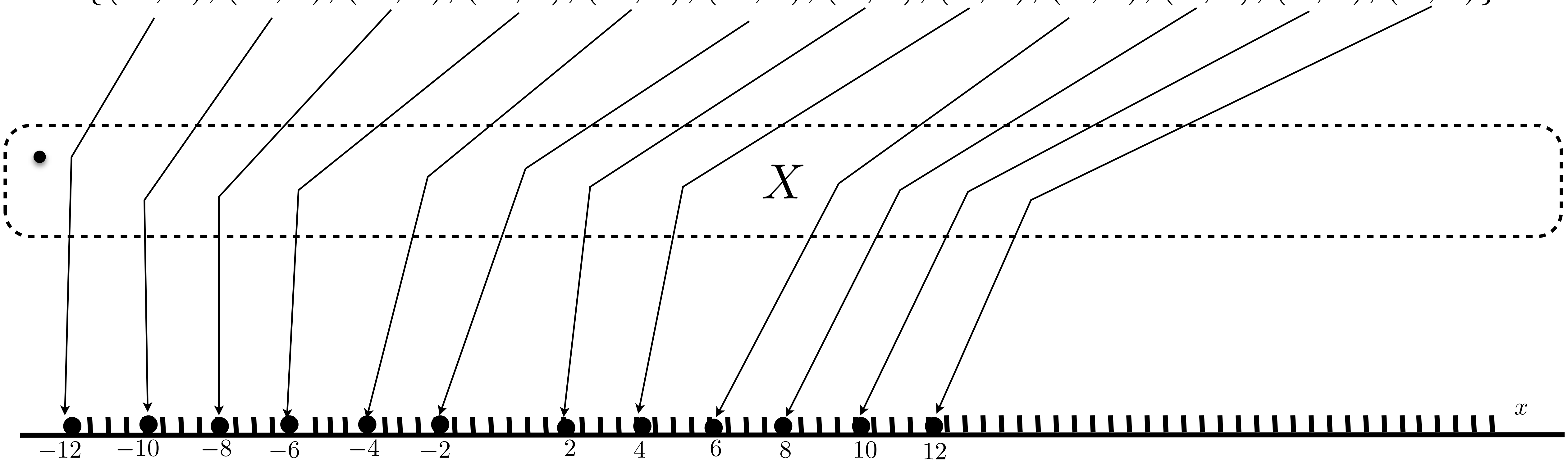
$$\binom{n}{r} = \binom{n}{n-r}$$

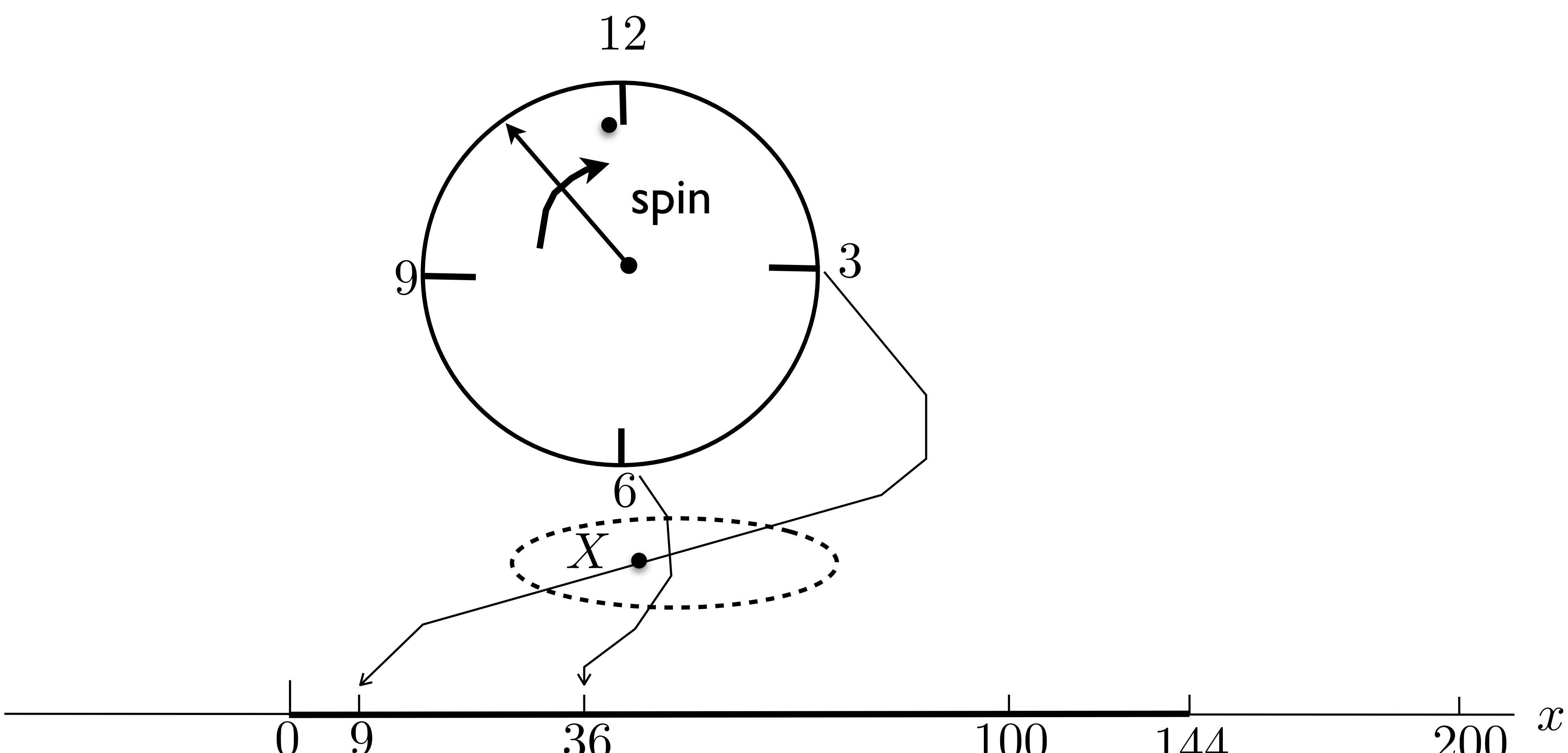
The Random Variable (RV)

- A real random variable is defined as
  - a real function of the elements of a sample space Ω
- Represent a random variable by a capital letter such as $W,\ X,\ \text{or}\ Y$ and any particular value of the random variable by a lowercase letter such as $w,\ x,\ \text{or}\ y.$
- Thus, given an experiment defined by a sample space Ω with elements ω, we assign to every ω a real number $X(\omega)$
  - according to some rule and call $X(\omega)$ a random variable.

- Example

$\Omega = \{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$





Conditions for a Function to be a Random

- First condition
  - The set $\{X \leq x\}$ shall be an event for any real number x .
  - The probability of this event, denoted by $P\{X \leq x\}$ is equal to the sum of the probabilities of all the elementary events corresponding to $\{X \leq x\}$.

- Probabilities of the events $\{X = \infty\}$ and $\{X = -\infty\}$

$$P\{X = \infty\} = 0 \text{ and } P\{X = -\infty\} = 0$$

- Probabilities of the events $\{X \leq \infty\}$

$$P\{X \leq \infty\} = 1$$

Categorization of Random Variables

- Continuous random variable
- Discrete random variable
- Mixed random variable

Bernoulli Trials

- There exist two outcomes in the experiment.
  - Example:
    - binary bit 1 or 0 is generated
    - Head or tail
- Denote each of two outcomes as A and $\bar{A}$
- Repeat experiments N times and A is observed k times out of the N trials.
  - Such repeated experiments are called *Bernoulli trials*.
- Probability

$$P(A) = p \text{ then } P(\bar{A}) = 1 - p$$

- k times out of N trials for the event A
  - one particular sequence is k times of A and $N-k$ times of $\bar{A}$ and its probability is

$$\underbrace{P(A)P(A)\cdots P(A)}_{k \text{ terms}}\underbrace{P(\bar{A})P(\bar{A})\cdots P(\bar{A})}_{N-k \text{ terms}} = p^k(1-p)^{N-k}$$

- Probability that A occurs exactly k times

$$P(A \text{ occurs exactly } k \text{ times}) = \binom{N}{k} p^k (1-p)^{N-k}$$

Distribution Function

- Cumulative distribution function (CDF)

$$F_X(x) = P\{X \leq x\}$$

- Properties of CDF

$$\textbf{(1)}\ \ F_X(-\infty) = 0$$

$$\textbf{(2)}\ \ F_X(\infty) = 1$$

$$\textbf{(3)}\ \ 0 \leq F_X(x) \leq 1$$

$$\textbf{(4)}\ \ F_X(x_1) \leq F_X(x_2), \qquad \text{if } x_1 < x_2$$

$$\textbf{(5)}\ \ P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1)$$

$$\textbf{(6)}\ \ F_X(x^+) = F_X(x)$$

- If the values of x_i , we may write $F_X(x)$

$$F_X(x) = \sum_{i=1}^{N} P\{X = x_i\} u(x - x_i)$$

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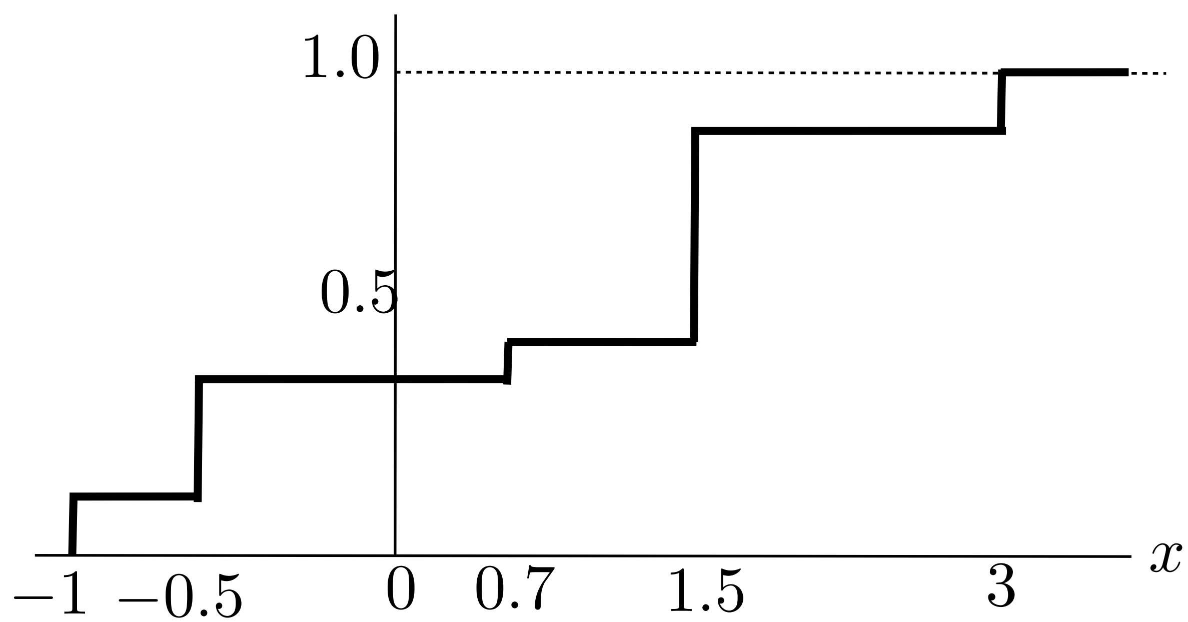
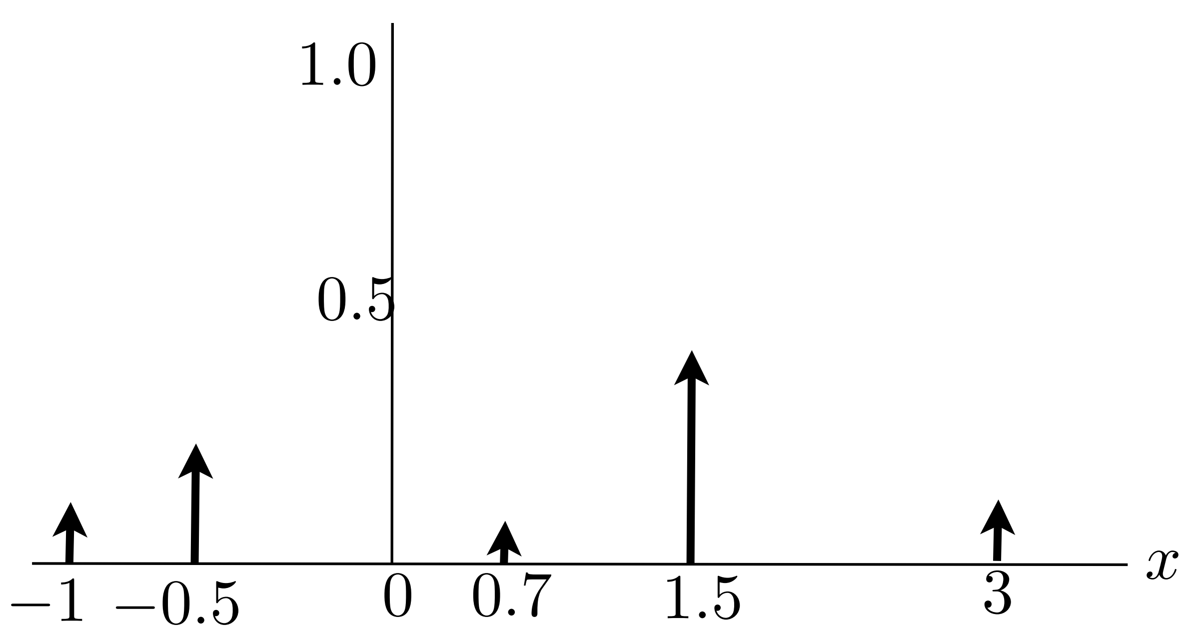




Probability Density Function (PDF)

- PDF is defined as the derivative of CDF.

$$f_X(x) = \frac{dF_X(x)}{dx}$$

- Properties of PDF

$$\textbf{(1)}\ \ 0 \le f_X(x) \qquad \text{all } x$$

$$\textbf{(2)}\ \ \int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

$$\textbf{(3)}\ \ F_X(x) = \int_{-\infty}^{x} f_x(\zeta)\, d\zeta$$

$$\textbf{(4)}\ \ P\{x_1 < X \le x_2\} = \int_{x_1}^{x_2} f_X(x)\, dx$$

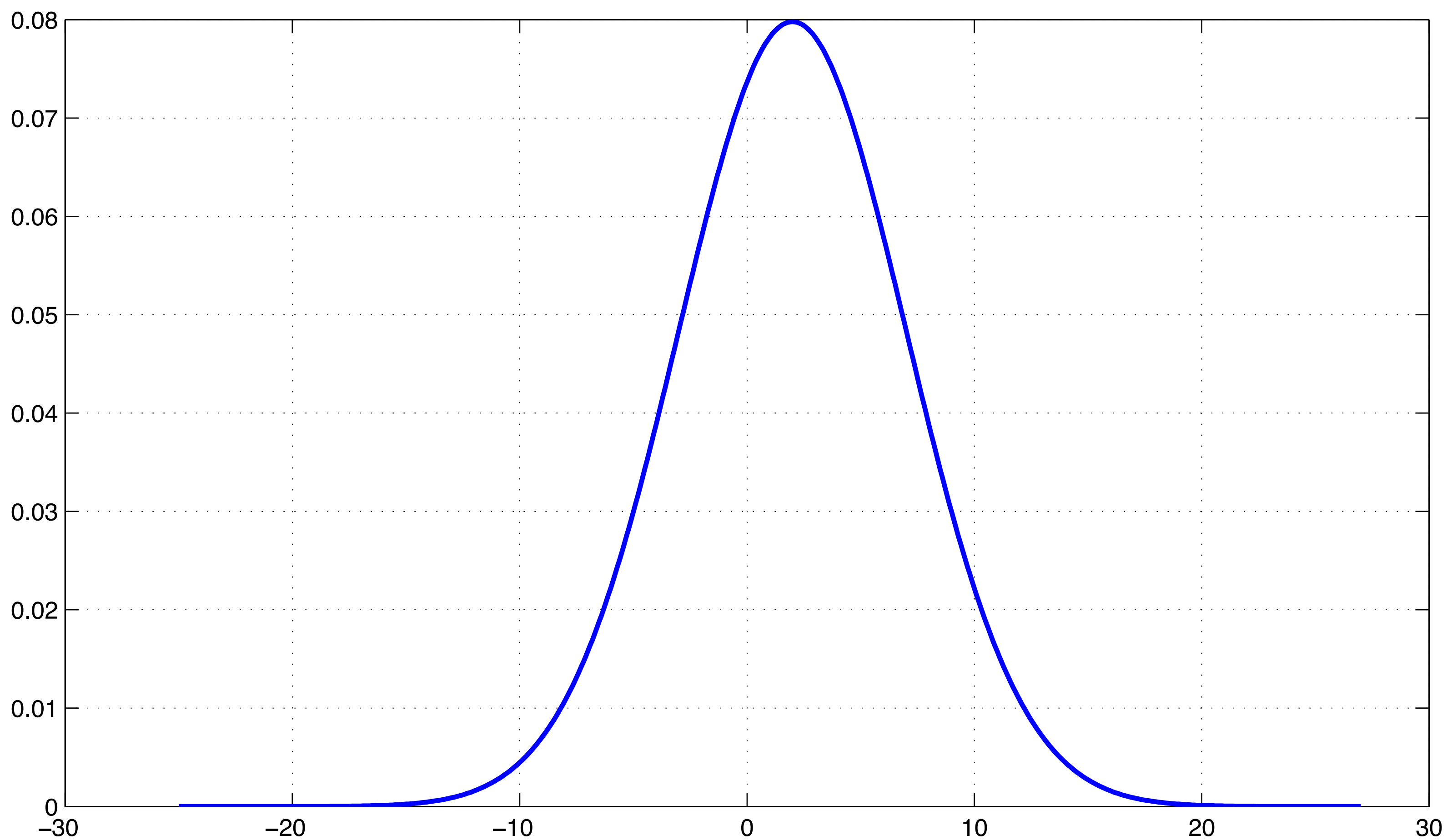
Gaussian Random Variable

- A random variable X is called gaussian if its density function has the form

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-m)^2}{2\sigma_x^2}\right] \qquad \sigma_X > 0 \text{ and } -\infty < m < \infty$$

- Example

$$m = 2 \text{ and } \sigma = 5$$

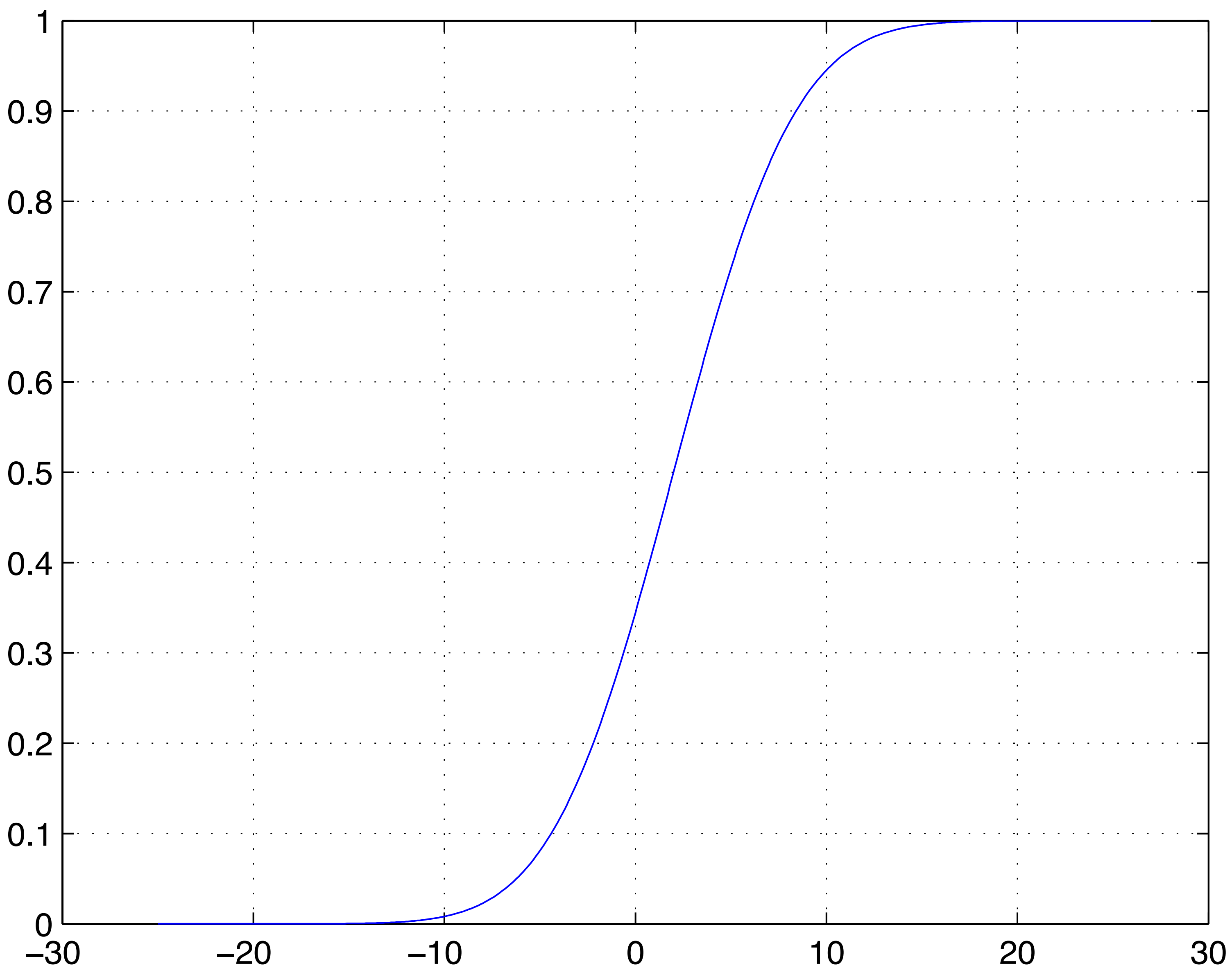


CDF

$$F_X(x) = \Pr[X \le x] = \int_{-\infty}^{x} f_X(\zeta)\, d\zeta = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\zeta - m)^2}{2\sigma_X^2}\right] d\zeta$$

$$= \int_{-\infty}^{\frac{(x-m)}{\sigma_X}} \exp\left[-\frac{t^2}{2}\right] dt = 1 - \int_{\frac{(x-m)}{\sigma_X}}^{\infty} \exp\left[-\frac{t^2}{2}\right] dt$$

$$= 1 - Q\left(\frac{x-m}{\sigma_X}\right)$$



where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt$$

Some Special Functions

- Q-function

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt$$

- Error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2}\, dt$$

  - Properties of error function
    - symmetry relation: $\mathrm{erf}(-x) = -\mathrm{erf}(x)$
    - As x approaches infinity, $\mathrm{erf}(x)$ approaches unity; that is,

$$\frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t^2}\, dt = 1$$

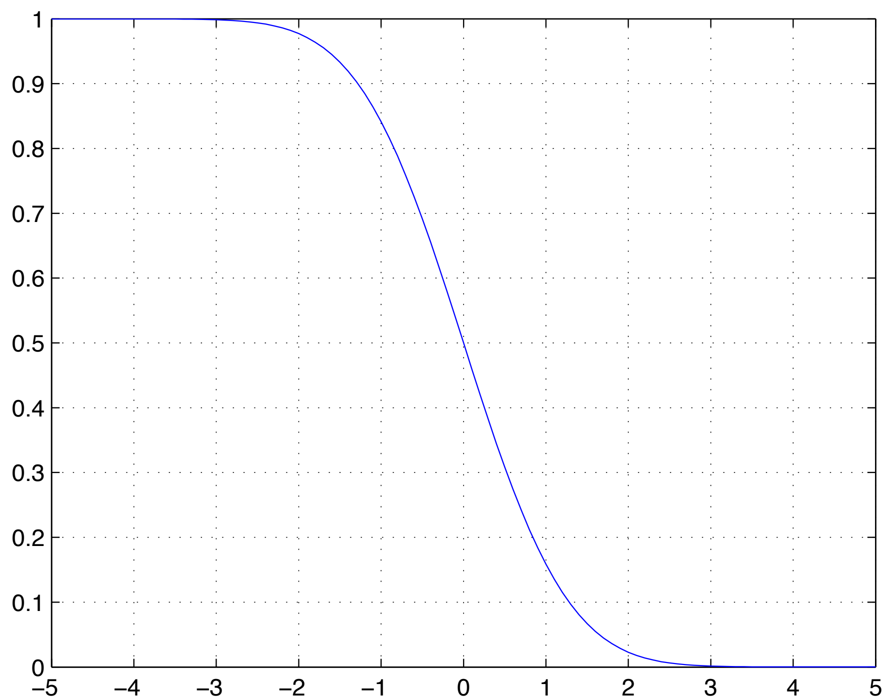
    - Complementary error function

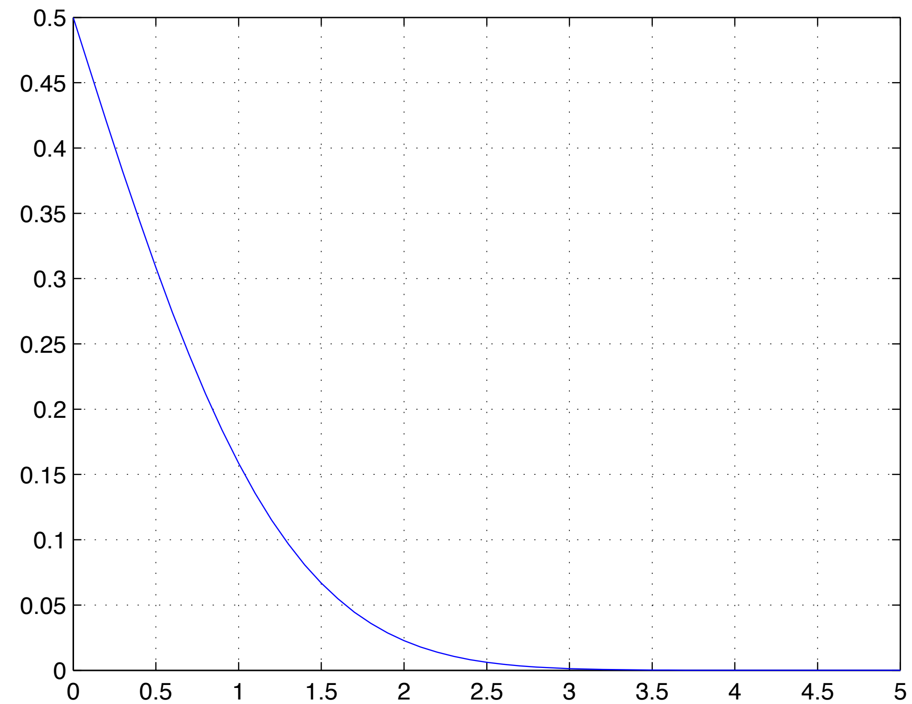
$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2}\, dt = 1$$

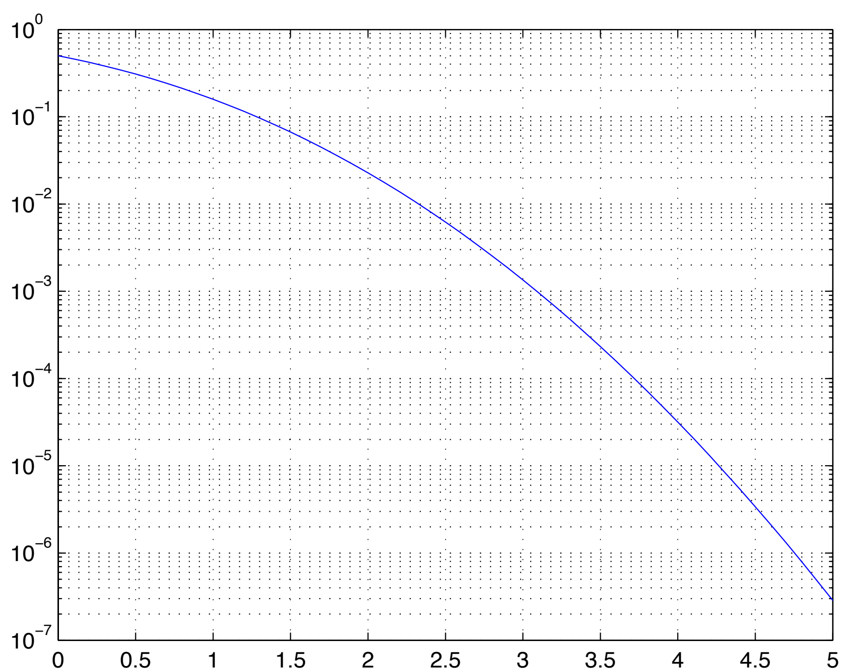
- Relation between Q and erfc functionss

$$
\begin{aligned}
Q(x) &= \frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right) \\
\operatorname{erfc}(x) &= 2Q(\sqrt{2}x)
\end{aligned}
$$

Q-function Plot







semilog plot

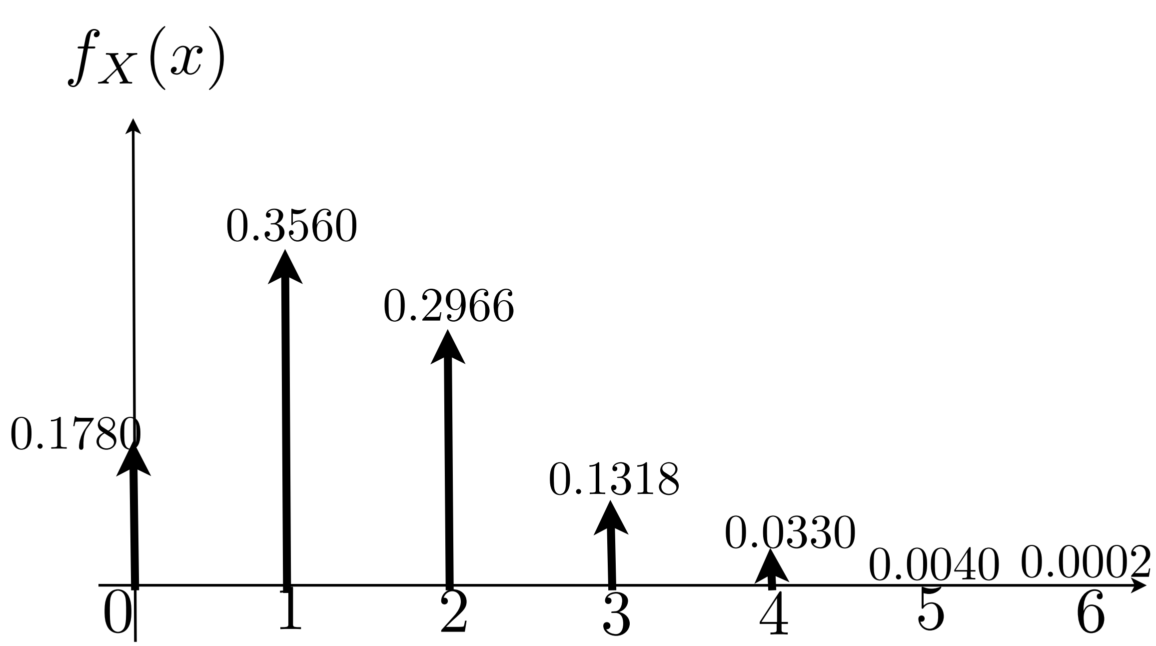
Binomial Distribution and Density

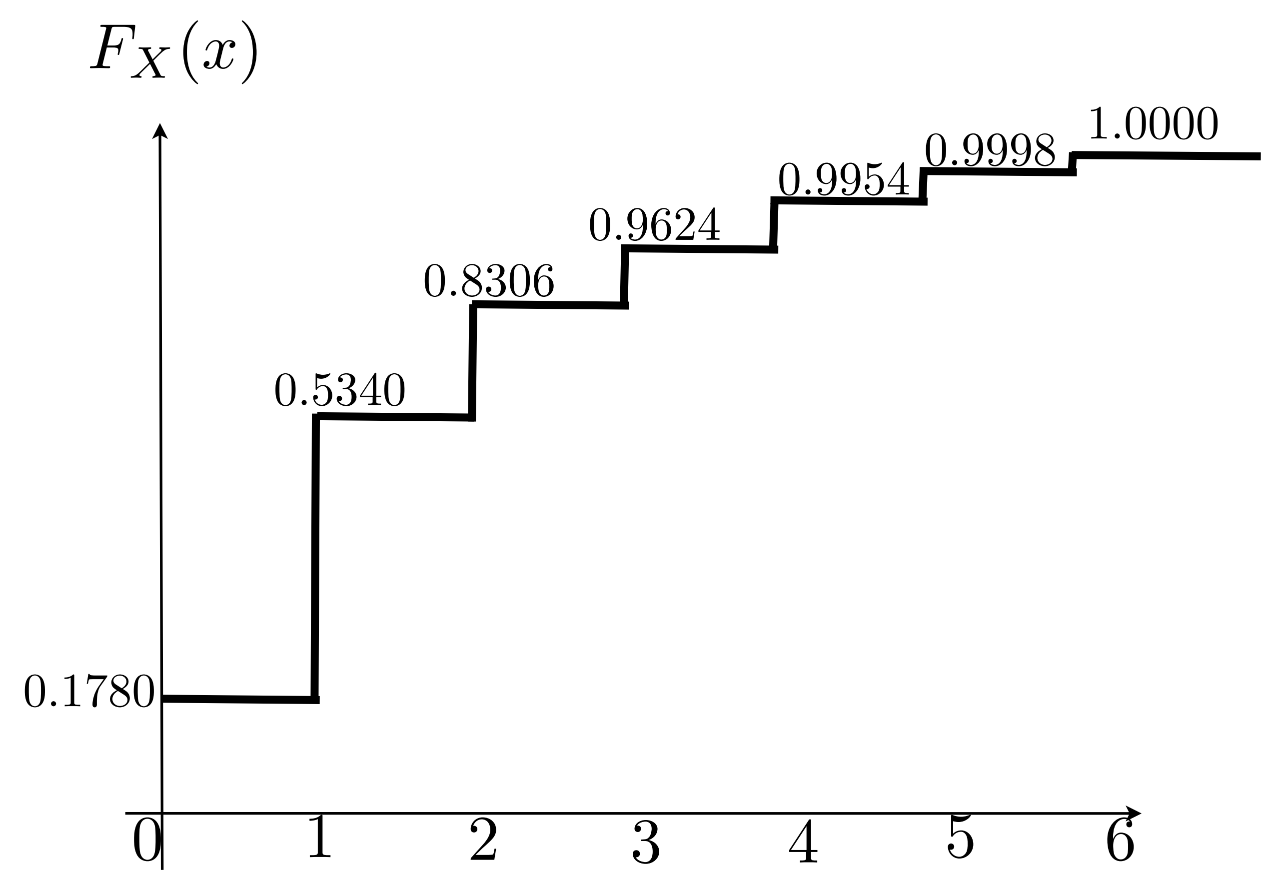
- Let $0 < p < 1,\ \text{and}\ N = 1, 2, \ldots$. Then,
  - PDF

$$f_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \delta(x-k)$$

  - CDF

$$F_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} u(x-k)$$





Uniform Distribution and Density

- PDF

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{elsewhere} \end{cases}$$

- CDF

$$F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a}, & a \le x < b \\ 1, & b \le x \end{cases}$$

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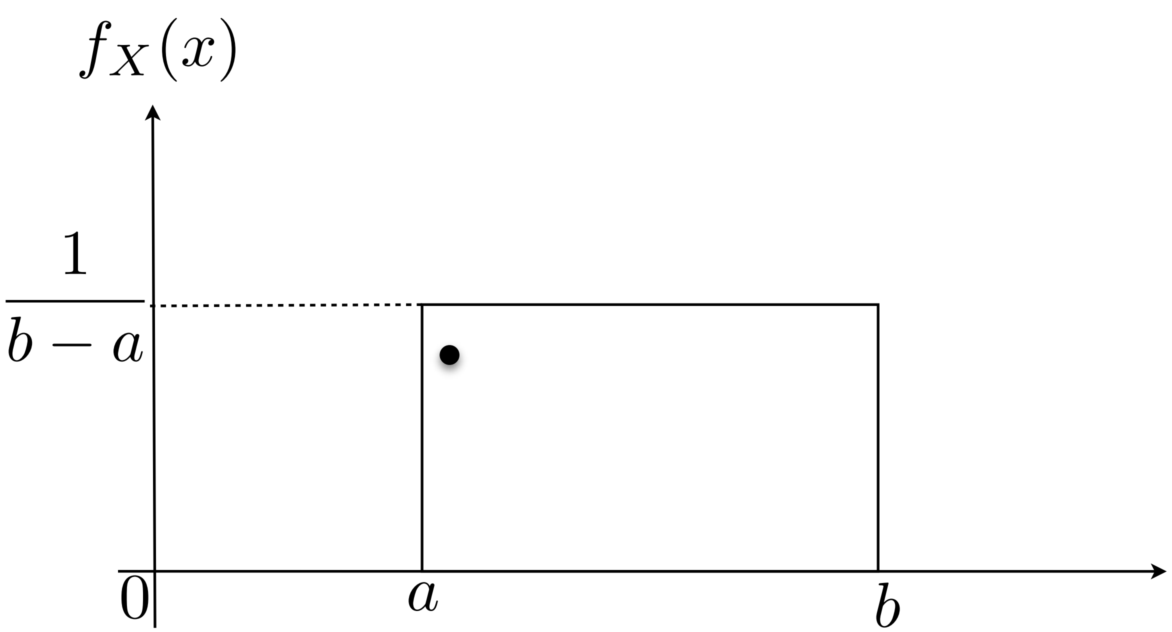
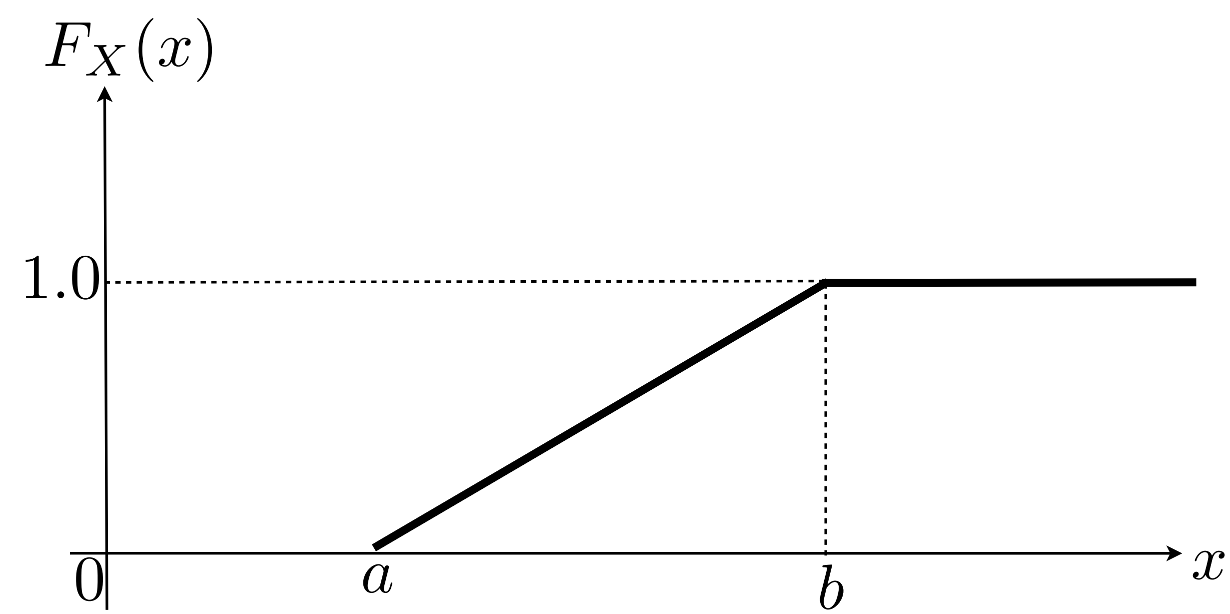




Uniform Random Samples in Matlab

- In matlab, "rand(N)" generates the N random samples distributed uniformly between zero and one.

- For example,

```
u=rand(100);        % generates 10 uniformly distributed random samples

u =    0.8147    0.9058    0.1270    0.9134    0.6324    0.0975    0.2785    0.5469    0.9575    0.9649
```

- Binary random sample generation with probability of half for zero and one, respectively.

```
u=rand(1,1);

if (u<0.5)

        b=0;
```

Exponential Distribution and Density

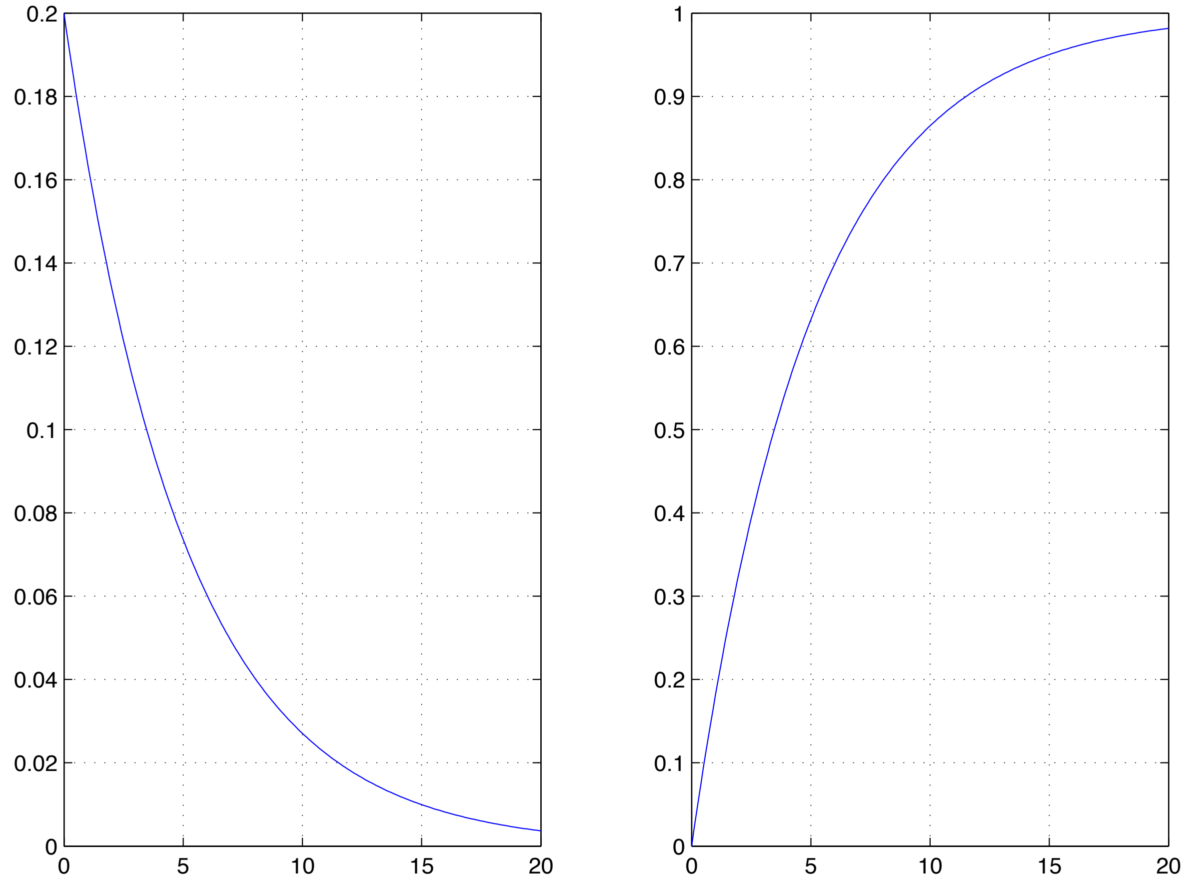
- PDF

$$f_X(x) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \quad \text{for } x \geq 0$$

- CDF

$$F_X(x) = 1 - e^{-\frac{x}{\lambda}} \quad \text{for } x \geq 0$$

- Example for $\lambda = 5$



Rayleigh Distribution and Density

- PDF

$$f_X(x) = \frac{2}{\lambda} x e^{-\frac{x^2}{\lambda}} \qquad \text{for } x \geq 0$$

- CDF

$$F_X(x) = 1 - e^{-\frac{x^2}{\lambda}} \quad \text{for } x \geq 0$$

- Example for $\lambda = 5$