Centroid

For non-uniform points

Definition

- The centroid (or geometric center, center of mass) is the point which the region could be perfectly balanced on the tip of a pencil or a finger, informally.

- As a basic knowledge, centroid of a triangle is the point of intersection of lines joining each vertex with the midpoint of the opposite side.

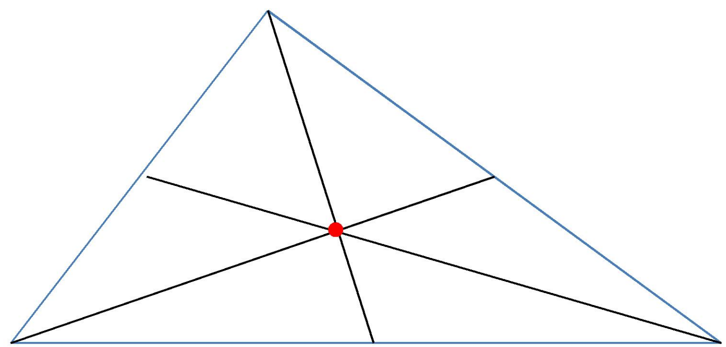


Fig. Centroid of a triangle

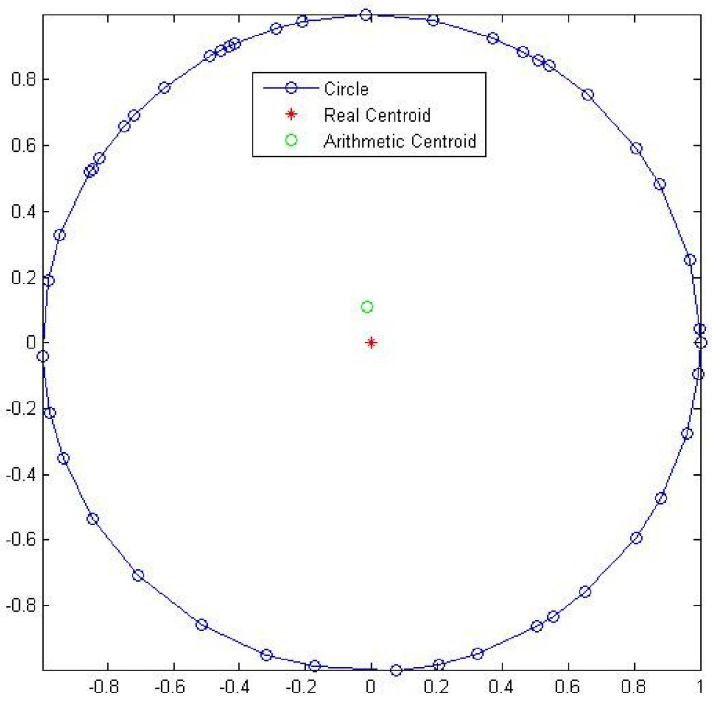
Definition

- Formally, the centroid is considered as the arithmetic mean (or average) point of all points of a two-dimensional shape.

- In this lecture, we focus on calculating the centroid numerically when the centroid is not equal to the numerical arithmetic mean point.

- In shear flow simulation, it happens when the boundary points of drop is not located well.

Numerical Centroid



- For non-uniform boundary points, the real centroid and arithmetic centroid is not equal numerically as shown in the left figure.

- Arithmetic centroid when M is the total mass :

$$(\bar{x}, \bar{y}) = \left(\frac{\sum x_i}{M}, \frac{\sum y_i}{M} \right)$$

Numerical Centroid

- The reason why they are different is the difference between an average and a weighted average.

- If the points are located very well, they are same. However, it is too hard to be in general.

- Centroid (using weighted average) :

$$\bar{x} = \frac{\int x f(x) dx}{\int f(x) dx}$$

- Here, we focus on the constant mass density, i.e. f = 1.

Theorem

- The numerical centroid (Cx, Cy) is given by

$$C_x = \frac{1}{6A} \sum_{i=0}^{n-1} (x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$C_y = \frac{1}{6A} \sum_{i=0}^{n-1} (y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

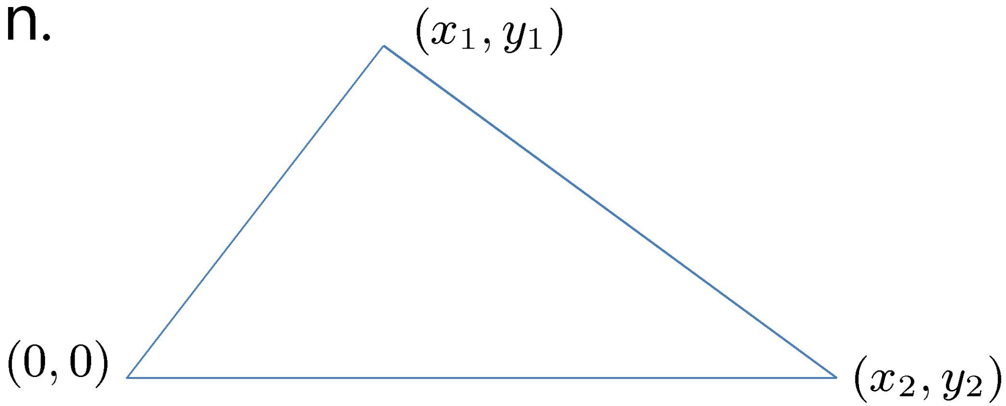
when

$$A = \frac{1}{2} \sum_{i=0}^{n-1} (x_i y_{i+1} - x_{i+1} y_i)$$

This is a discritized version of the weighted average in the last slide.

Proof

- It is based on the centroid on the polygon divided by some triangles. So, we prove for just a triangle here and it can be generalized to other n-gons by summation.



- A is an area of triangle which can be driven by the curl of (0, 0), (x1, y1) and (x2, y2).

Proof

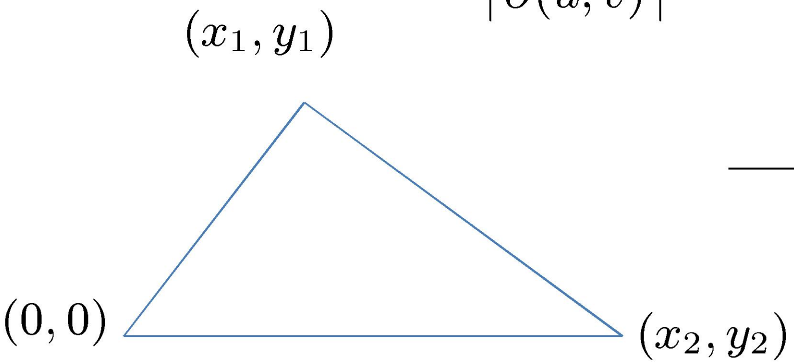
- At first, we translate our triangle to more simple one.

- Consider (u, v) such that

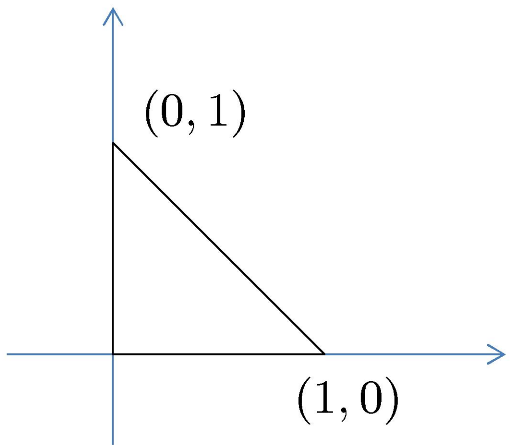
$$x = x_1 u + x_2 v \qquad (x_1, y_1) \to (1, 0)$$
$$\longrightarrow \qquad (x_2, y_2) \to (0, 1)$$
$$y = y_1 u + y_2 v \qquad (0, 0) \to (0, 0)$$

and

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = |x_1 y_2 - x_2 y_1| = A$$



$$\longrightarrow$$



Proof

- So,

$$\int xf(x)dx = \int_{x_2}^{x_1} \int_{y_2}^{y_1} xdxdy$$

$$= \int_0^1 \int_0^{1-v} (x_1u + x_2v)Adudv$$

$$= A\int_0^1 \left[\frac{x_1}{2}(1-v)^2 + x_2(1-v)v\right] dv$$

$$= A\left[\left(-\frac{x_1}{6}\right)(1-v)^3 + x_2\left(\frac{1}{2}v^2 - \frac{1}{3}v^3\right)\right]_0^1$$

$$= \frac{A}{6}(x_1 + x_2)$$

Proof (complete)

- From the result of slide number 6 and 9,

$$\bar{x} = \frac{\int x f(x) dx}{\int f(x) dx}$$

$$= \frac{\frac{A}{6}(x_1 + x_2)}{A}$$

$$= \frac{1}{6A}(x_1 + x_2)(x_1 y_2 - x_2 y_1)$$

- We can get the mean of y also, by same derivation and it can be expanded to non-gon containing more points by simple summation.